MODEL ANSWERS TO HWK #1

1. (i) Clear.
(ii) The equaliser of f_1 and $f_2\colon X \longrightarrow Y$ is the set

$$\{\, x \in X \mid f_1(x) = f_2(x) \,\},$$

together with its natural inclusion into X.
(iii) Suppose that $\mathcal{C}$ admits equalisers. Let $f\colon X \longrightarrow B$ and $g\colon Y \longrightarrow B$ be two morphisms. Then there are two morphisms $p\colon X \times Y \longrightarrow B$ (respectively q), the composition of projection down to X (respectively projection down to Y) and then f (respectively g) as appropriate. Let E be the equaliser of p and q. Then E maps to $X \times Y$, whence it maps to X and Y, via either projection, and these two morphisms become equal when composed with f and g. Now suppose that Z maps to both X and Y over B. Then it maps to $X \times Y$ and the composition of this map with either p or q is the same as the original map from Z to B. It follows that Z maps to E, by the universal property of the equaliser. But then E satisfies the universal property of the fibre product.
Now suppose that $\mathcal{C}$ admits fibre products. If f and $g\colon X \longrightarrow Y$ are two morphisms, then we get a morphism $X \longrightarrow Y \times Y$, by definition of the product. Note that there is also a morphism $\delta\colon Y \longrightarrow Y \times Y$ induced by the identity on both factors. Let $E = X \underset{Y \times Y}{\times} Y$. Then E maps to X and composing this map with either f or g is the same. Suppose that Z maps to X, such that the composition with f or g is the same. Then Z maps to X and its maps to Y over $Y \times Y$. So Z maps to E, by the universal property of the fibre product. But then E satisfies the universal property of the equaliser.
2. 2.7. Since K is a field, it has a unique prime ideal, and so $\operatorname{Spec} K$ certainly has only one point, and the structure sheaf is represented by K itself. To give a morphism of $\operatorname{Spec} K$ to X, we certainly have to pick out a point $x \in X$. But then, by definition of a scheme, there is an induced morphism of local rings,

$$\mathcal{O}_{X,x} \longrightarrow K$$

But this is equivalent to a ring homomorphism, which sends the maximal ideal m_x to a point, which in turn is equivalent to giving an inclusion of the residue field of x into K.

2.16. (a) Suppose that $x \in U$. As U is open,

$$\mathcal{O}_{U,x} \simeq \mathcal{O}_{X,x},$$

and the rest is clear.

(b) As X is compact there is an open cover $\{U_1, U_2, \dots, U_k\}$ of X by finitely many affines. By our answer to part (a), $X_f \cap U_i = U_{f_i}$, where f_i is the restriction of f to U_i. As a is zero on X_f, its restriction a_i to U_{f_i} is zero. As $U_i = \operatorname{Spec} A_i$ is affine, it follows that $U_{f_i} = \operatorname{Spec}(A_i)_{f_i}$. In particular $f_i^{n_i} a_i = 0$, for some $n_i \in \mathbb{N}$. As we have a finite cover, we may assume that $n = n_i$ is independent of i. We may also assume $n > 0$. Since the restriction of $f^n a \in \Gamma(X, \mathcal{O}_X)$ to each set U_i of the open cover $\{U_1, U_2, \dots, U_k\}$ is zero, it follows that $f^n a$ is zero.

(c) Let b_i be the restriction of b to U_i and let f_i be the restriction of f to U_i. As U_i is affine and $X_f \cap U_i = U_{f_i}$ by part (a), we may lift $f_i^{n_i} b_i$ to c_i on U_i. Now $c_i - c_j$ restricts to zero on $U_{ij} \cap X_f$. As we are assuming that U_{ij} is compact, it follows that $c_i - c_j$ restricts to zero on the whole of U_{ij}, by our answer to part (b). But then there is a section c on the whole of X which restricts to c_i on U_i. The axioms for a sheaf also imply that c is a lift of $f^n b$.

(d) Note first that X is compact as it has a finite cover by open affines, which are always compact.

Consider the natural restriction map

$$A = \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X_f, \mathcal{O}_{X_f}).$$

As f is sent to a unit, there is a natural map

$$A_f \longrightarrow \Gamma(X_f, \mathcal{O}_{X_f}).$$

The answer to part (b) proves that this map is injective and the answer to part (c) that it is surjective. Hence this map is an isomorphism.

2.17 (a) The map on topological spaces is surely a homeomorphism under these circumstances. It suffices, then, to check that the map on structure sheaves is an isomorphism. As this may be checked on stalks, the result follows.

(b) If X is affine, just take $r = f = 1$.

Otherwise suppose that we have $f_1, f_2, \dots, f_r$ such that U_{f_i} is affine, where $f_1, f_2, \dots, f_r$ generate the unit ideal. Let $Y = \operatorname{Spec} A$. By (2.4) there is a morphism

$$f \colon X \longrightarrow Y,$$

induced by the identity map $A \longrightarrow A$. Let V_{f_i} be the open affine subset of Y where f_i is not zero. Then $f^{-1}(V_{f_i}) = U_{f_i}$ and both sets are affine. By our answer to (2.16.d), they are both isomorphic to $\operatorname{Spec} A_{f_i}$ and the induced map on A_{f_i} is the identity. So the morphism f is certainly an isomorphism over the open subset V_{f_i}. But since $f_1, f_2, \dots, f_r$ generate the unit ideal, these sets cover X and we are done by part (a).

Before we prove the next exercises, we recall a result that was proved implicitly in the lectures. Suppose that X is a scheme and that $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are two affine schemes. Then $U \cap V$ be covered by finitely many affine schemes which are simultaneously isomorphic to U_g and V_h, where $g \in A$ and $h \in B$.

3.1 It suffices to prove that $f^{-1}(V)$ is covered by open affines $U = \operatorname{Spec} A$ such that A is a finitely generated B-algebra. By the observation above we may assume that $Y = V = \operatorname{Spec} B$ and that we can cover Y by finitely many open affine subsets $V_i = U_{f_i}$, where $f^{-1}(V_i)$ can be covered by open affines which are the spectra of finitely generated B_{h_i}-algebras. For each i, pick $U_i = \operatorname{Spec} A_i$ lying over V_i where A_i is a finitely generated B_i-algebra.

Let U be the union of $U_1, U_2, \dots, U_k$. As sets of this form cover $f^{-1}(Y)$ it suffices to prove that U is an open affine which is the spectrum of a finitely generated B-algebra. It is clear that U is open.

Let g_i be the image of f_i in $A = \Gamma(U, \mathcal{O}_U)$. Then U_i is the locus where g_i is not equal to zero. $g_1, g_2, \dots, g_k$ generate the unit ideal of A, as $f_1, f_2, \dots, f_k$ generate the unit ideal of B. It follows by (2.17), that U is affine and it suffices to prove that A is a finitely generated B-algebra.

So now we are reduced to the following problem in algebra. Let B be an A-algebra, and let $f_1, f_2, \dots, f_k$ generate the unit ideal. Suppose that g_i is the image of f_i and suppose that $A_i = A_{g_i}$ is a finitely generated $B_i = B_{f_i}$-algebra. Then A is a finitely generated B-algebra.

We now prove this result in commutative algebra. To this end, pick generators $c_{i1}, c_{i2}, \dots, c_{il_i}$ of A_i over B_i. Then each c_{ij} has the form a_{ij}/g_i^n, where we may assume that n is constant, as we have only finitely many indices. I claim that a_{ij}, for every i and j, generates A over B. Pick $a \in A$. Then if $\phi_i \colon A \longrightarrow A_i$ is the natural map, we have

$$\phi_i(a) = p(c_{ij}),$$

for some for some polynomial p, with coefficients in B_i. Clearing denominators, we then have

$$g_i^N a = q(a_{ij}),$$

for some polynomial q, with coefficients in A_i. We may write

$$\sum h_i g_i^N = 1,$$

for some h_i. But then

$$
\begin{aligned}
a &= \sum h_i g_i^N a, \\
&= \sum h_i \left(\sum_j q(a_{ij}) \right),
\end{aligned}
$$

as required.

3.2 The key observation is that a scheme is compact iff it is the finite union of affine subschemes. Indeed, if X is a scheme, then it is union of open affine subschemes, and if X is compact, then finitely many cover. Conversely, any affine scheme is compact, and the finite union of compact sets is always compact.

So now suppose that $f\colon X \longrightarrow Y$ is a compact morphism, and let V be an affine subset. Using the argument just before (3.1) we may assume that $Y = V$. Let V_i be an open affine cover of Y such that $f^{-1}(V_i)$ is compact. As Y is affine we may assume that this cover is finite. But then $f^{-1}(Y)$ is compact, as it is a finite union of compact subsets.

3.3 (a) Clear, from the first paragraph of (3.2).

(b) Simply apply (3.1) and (3.2).

(c) By now standard tricks, we can reduce this problem to showing that if a B-algebra A contains elements $f_1, f_2, \dots, f_k$ which generate the unit ideal and A_{f_i} is a finitely generated B-algebra, then so is A. But this is easily implied by part of the proof of (3.1).

3.4 Follows almost exactly the same proof as (3.1), and (3.3.c). We are reduced to proving that if A is a B-algebra and $f_1, f_2, \dots, f_k$ are elements of B which generate the unit ideal, such that $A_i = A_{g_i}$ is a finitely generated $B_i = B_{f_i}$-module, where g_i is the image of f_i, then A is a finitely generated B-module.

Repeating the argument given in (3.1), we are given $a_{ij} \in A$ whose images c_{ij} under ϕ_i generate A_i as a B_i-module. As before this implies that

$$g_i^N a = \sum_j b_{ij} a_{ij},$$

for some $b_{ij} \in B$. It is then easy to see that we may write a as a linear combination of a_{ij}, so that the a_{ij} generate A as a B-module.