9. Varieties as schemes

Now we turn to the definition of projective schemes.

The definition mirrors that for affine schemes. First we start with a graded ring S,

$$S = \bigoplus_{d \in \mathbb{N}} S_d.$$

We set

$$S_+ = \bigoplus_{d > 0} S_d,$$

and we let $\operatorname{Proj} S$ denote the set of all homogeneous prime ideals of S, which do not contain S_+. We put a topology on $\operatorname{Proj} S$ analogously to the way we put a topology on $\operatorname{Spec} S$; if $\mathfrak{a}$ is a homogeneous ideal of S, then we set

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

The Zariski topology is the topology where these are the closed sets. If $\mathfrak{p}$ is a homogeneous prime ideal, then $S_{(\mathfrak{p})}$ denotes the elements of degree zero in the localisation of S at the set of homogenous elements which do not belong to $\mathfrak{p}$. We define a sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Proj} S$ by considering, for an open set $U \subset X$, all functions

$$s \colon U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})},$$

such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, which are locally represented by quotients. That is given any point $\mathfrak{q} \in U$, there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a and f in S of the same degree, such that for every $\mathfrak{p} \in V$, $f \notin \mathfrak{p}$ and $s(\mathfrak{p})$ is represented by the class of $a/f \in S_{(\mathfrak{p})}$.

Proposition 9.1. *Let S be a graded ring and set $X = \operatorname{Proj} S$.*

(1) For every $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $S_{(\mathfrak{p})}$.

(2) For any homogeneous element $f \in S_+$, set

$$U_f = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \,\}.$$

Then U_f is open in $\operatorname{Proj} S$, these sets cover X and we have an isomorphism of locally ringed spaces

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq \operatorname{Spec} S_{(f)}.$$

where $S_{(f)}$ consists of all elements of degree zero in the localisation S_f.

In particular $\operatorname{Proj} S$ is a scheme.

Proof. The proof of (1) follows similar lines to the affine case and is left as an exercise for the reader. $U_f = X - V(\langle f\rangle)$ and so U_f is certainly open and these sets certainly cover X. We are going to define an isomorphism

$$(g, g^{\#})\colon (U_f, \mathcal{O}_X|_{U_f}) \longrightarrow \operatorname{Spec} S_{(f)}.$$

If $\mathfrak{a}$ is any homogeneous ideal of S, consider the ideal $\mathfrak{a}S_f \cap S_{(f)}$. In particular if $\mathfrak{p}$ is a prime ideal of S, then $\phi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$ is a prime ideal of $S_{(f)}$. It is easy to see that ϕ is a bijection. Now $\mathfrak{a} \subset \mathfrak{p}$ iff

$$\mathfrak{a}S_f \cap S_{(f)} \subset \mathfrak{p}S_f \cap S_{(f)} = \phi(\mathfrak{p}),$$

so that ϕ is a homeomorphism. If $\mathfrak{p} \in U_f$ then $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\phi(\mathfrak{p})}$ are naturally isomorphic. This induces a morphism $g^{\#}$ of sheaves which is easily seen to be an isomorphism. □

Definition 9.2. *Let R be a ring.* ***Projective n-space over R****, denoted $\mathbb{P}^n_R$, is the proj of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

Note that $\mathbb{P}^n_R$ is a scheme over $\operatorname{Spec} R$.

Definition-Lemma 9.3. *If X is a topological space, then let $t(X)$ be the set of irreducible closed subsets of X. Then $t(X)$ is naturally a topological space and if we define a map $\alpha\colon X \longrightarrow t(X)$ by sending a point to its closure then α induces a bijection between the closed sets of X and $t(X)$.*

Proof. Observe that

- If $Y \subset X$ is a closed subset, then $t(Y) \subset t(X)$,
- if Y_1 and Y_2 are two closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, and
- if Y_α is any collection of closed subsets, then $t(\cap Y_\alpha) = \cap t(Y_\alpha)$.

The defines a topology on $t(X)$ and the rest is clear. □

Theorem 9.4. *Let k be an algebraically closed field. Then there is a fully faithful functor t from the category of varieties over k to the category of schemes. For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the sheaf of regular functions is the restriction of the structure sheaf to the set of closed points.*

Proof. We will show that $(t(V), \alpha_*\mathcal{O}_V)$ is a scheme, where $\mathcal{O}_V$ is the sheaf of regular functions on V. As any variety has an open affine cover, it suffices to prove this for an affine variety, with coordinate ring

A. Let X be the spectrum of A. We are going to a define a morphism of locally ringed spaces,

$$\beta = (f, f^{\#}) \colon (V, \mathcal{O}_V) \longrightarrow (X, \mathcal{O}_X).$$

If $p \in V$, then let $f(p) = m_p \in X$ be the maximal ideal of elements of A vanishing at p. By the Nullstellensatz, f induces a bijection between the closed points of X and the points of V. It is easy to see that f is a homeomorphism onto its image. Now let $U \subset X$ be an open set. We need to define a ring homomorphism

$$f^{\#}(U) \colon \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_V(f^{-1}(U)).$$

Let $s \in \mathcal{O}_X(U)$. We want to define $r = f^{\#}(U)(s)$. Pick $p \in U$. Then we define $r(p)$ to be the image of $s(m_p) \in A_{m_p}$ inside the quotient

$$A_{m_p}/m_p \simeq k.$$

It is easy to see that r is a regular function and that $f^{\#}(U)$ is a ring isomorphism. As the irreducible subsets of V are in bijection with the prime ideals of A, it follows that $(X, \mathcal{O}_X)$ is isomorphic to $(t(V), \alpha_*\mathcal{O}_V)$, and so the latter is an affine scheme.

Note that there is a natural inclusion

$$k \subset A,$$

which associates to a scalar the constant function on V. But then X is a scheme over $\operatorname{Spec} k$. It is easy to check that t is fully faithful. □

Before we check that projective morphisms are proper, let's do a warm up case:

Theorem 9.5. *Let $f \colon \Delta^* \longrightarrow \mathbb{P}^n$ be a meromorphic map of the punctured disk into projective space over $\mathbb{C}$.*

Then f extends to a holomorphic map $g \colon \Delta \longrightarrow \mathbb{P}^n$.

Proof.

$$f(z) = [f_0(z) : f_1(z) : \cdots : f_n(z)].$$

Each meromorphic function $f_i(z) = z^{m_i} h_i(z)$, where m_i is an integer and $h_i(z)$ is holomorphic, $h_i(0)$ is non-zero. Let m be the minimum of $m_0, m_1, \ldots, m_n$. Then

$$g_i(z) = z^{-m} f_i(z) = z^{m_i - m} h_i(z)$$

is holomorphic and at least one of $g_i(0)$ is non-zero. On the other hand

$$\begin{aligned} g(z) &= [g_0(z) : g_1(z) : \cdots : g_n(z)] \\ &= [z^{-m} f_0(z) : z^{-m} f_1(z) : \cdots : z^{-m} f_n(z)] \\ &= f(z), \end{aligned}$$

whenever z is non-zero, so that $g\colon \Delta \longrightarrow \mathbb{P}^n$ is a holomorphic map extending f. ☐

Definition 9.6. *Let $\pi\colon X \longrightarrow S$ be a morphism.*

We say that π is a ***projective morphism*** *if it can be factored into a closed embedding $i\colon X \longrightarrow \mathbb{P}^n_S$ and the projection morphism $\mathbb{P}^n_S \longrightarrow S$.*

Theorem 9.7. *A projective morphism is proper.*

Proof. Since a closed immersion is of finite type, and using the results of §8, it suffices to prove that $X = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Now X is covered by open affines of the form $U_i = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$. Thus X is certainly of finite type over $\operatorname{Spec}\mathbb{Z}$. We check the valuative criteria. Suppose we have a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & \operatorname{Spec}\mathbb{Z}. \end{array}$$

Let $\xi_1 \in X$ be the image of the unique point of U. By induction on n, we may assume that ξ_1 is not contained in any of the $n+1$ standard hyperplanes, so that $\xi_1 \in U = \bigcap U_i$. Thus the functions x_i/x_j are all invertible on U.

There is an inclusion $k(\xi_1) \subset K$. Let f_{ij} be the image of x_i/x_j. Then

$$f_{ik} = f_{ij} f_{jk}.$$

Let $\nu\colon K \longrightarrow G$ be the valuation associated to R. Let $g_i = \nu(f_{i0})$ and pick k such that g_k is minimal. Then

$$\nu(f_{ik}) = g_i - g_k \geq 0.$$

Hence $f_{ik} \in R$. Define a ring homomorphism

$$\mathbb{Z}[x_0/x_k, x_1/x_k, \dots, x_n/x_k] \longrightarrow R \qquad \text{by sending} \qquad x_i/x_k \longrightarrow f_{ik}.$$

This gives a morphism $T \longrightarrow U_k$ and by composition a morphism $T \longrightarrow X$. ☐

Using this, we can finally characterise the image of the functor t.

Proposition 9.8. *Fix an algebraically closed field k. Then the image of the functor t is precisely the set of integral quasi-projective schemes, and the image of a projective variety is an integral projective scheme.*

In particular for every variety V, $t(V)$ is an integral separated scheme of finite type over k.

Proof. It only suffices to prove that every integral projective scheme Y is the image of a variety. Let Y be a closed subscheme of $\mathbb{P}^n_k$. Then the set of closed points V is a closed subset of the variety $\mathbb{P}^n$. If Y is irreducible, as V is dense in Y, it follows that V is irreducible. If Y is reduced, it is easy to see that $t(V) = Y$, since they have the same support and they are both reduced. □

Definition 9.9. *A* ***variety*** *is an integral separated scheme of finite type over an algebraically closed field. If in addition it is proper, then we say that is a* ***complete variety****.*