8. Prim and proper

Okay prim is not a property of schemes, but proper and separated are. In this section we want to extend the intuitive notions of being Hausdorff and compact to the category of schemes.

First we come up with a formal definition of both properties and then we investigate how to check the formal definitions in practice. We start with the definition of separated, which should be thought as corresponding to Hausdorff.

Definition 8.1. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. The* ***diagonal morphism*** *is the unique morphism $\Delta\colon X \longrightarrow X \underset{Y}{\times} X$, given by applying the universal property of the fibre product to the identity map $X \longrightarrow X$, twice.*

We say that the morphism f is ***separated*** *if the diagonal morphism is a closed immersion.*

Example 8.2. *Consider the line X, with a double origin, obtained by gluing together two copies of $\mathbb{A}^1_k$, without identifying the origins. Consider the fibre square over k, $X \underset{k}{\times} X$. This is a doubled affine plane, which has two x-axes, two y-axes and four origins. The diagonal morphism, only hits two of those four origins, whilst the closure contains all four origins.*

Proposition 8.3. *Every morphism of affine schemes is separated.*

Proof. Suppose we are given a morphism of affine schemes $f\colon X \longrightarrow Y$, where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then the diagonal morphism is given by,

$$A \underset{B}{\otimes} A \longrightarrow A \qquad \text{where} \qquad a \otimes a' \longrightarrow aa'.$$

As this is a surjective ring homomorphism, it follows that Δ is a closed immersion. □

Corollary 8.4. *$f\colon X \longrightarrow Y$ is separated if and only if the image of the diagonal morphism is a closed subset.*

Proof. One direction is clear. So suppose that the image of the diagonal morphism is closed. We need to prove that $\Delta\colon X \longrightarrow \Delta(X)$ is a homeomorphism and that $\mathcal{O}_{X \underset{Y}{\times} X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective. Consider the first projection $p_1\colon X \underset{Y}{\times} X \longrightarrow X$. As the composition $p_1 \circ \Delta$ is the identity, it is immediate that Δ is a homeomorphism onto its image.

To check surjectivity of sheaves, we may work locally. Pick $p \in X$ and an open affine neighbourhood $V \subset Y$ of the image $q \in Y$. Let U

be an open affine neighbourhood of p contained in the inverse image of V. Then $U \underset{V}{\times} U$ is an open affine neighbourhood of $\Delta(p)$, and by (8.3), $\Delta\colon U \longrightarrow U \underset{V}{\times} U$ is a closed immersion. But then the map of sheaves is surjective on stalks at p. □

The idea of how to characterise both properties (separated and proper), is based on the idea of probing with curves. After all, the clasic example of the doubled origin, admits an open immersion with two different extensions. Firstly, we need to work more locally than this, so that we want to work with local rings. However, our schemes are so general, that we also need to work with something more general than a curve. We recall some basic facts about valuations and valuation rings.

Definition 8.5. *Let K be a field and let G be a totally ordered abelian group. A* ***valuation*** *of K with values in G, is a map*

$$\nu\colon K - \{0\} \longrightarrow G,$$

such that for all x and $y \in K - \{0\}$ we have:

(1) $\nu(xy) = \nu(x) + \nu(y)$.
(2) $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Definition-Lemma 8.6. *If ν is a valuation, then the set*

$$R = \{\, x \in K \mid \nu(x) \geq 0 \,\} \cup \{0\},$$

is a subring of K, which is called the ***valuation ring*** *of ν. The set*

$$\mathfrak{m} = \{\, x \in K \mid \nu(x) > 0 \,\} \cup \{0\},$$

is an ideal in R and the pair $(R, \mathfrak{m})$ is a local ring.

Proof. Easy check. □

Definition 8.7. *A valuation is called a* ***discrete valuation*** *if $G = \mathbb{Z}$. The corresponding valuation ring is called a* ***discrete valuation ring****.*

Let X be a variety. There is essentially one way to get a discrete valuation of the function field of X.

Example 8.8. *Let X be a smooth variety, and let x be a point of X. Then every element of $K = K(X)$ is of the form $f/g \in \mathcal{O}_{X,x}$. We define the* ***order of vanishing*** *of f/g along x to be the difference*

$$a - b,$$

where $f \in m^a - m^{a+1}$ and $g \in m^b - m^{b+1}$. Then the order of vanishing defines a valuation ν of K.

Definition 8.9. *Let A and B be two local rings, with the same field of fractions. We say that B* ***dominates*** *A if $A \subset B$ and $m_A = m_B \cap A$. Now let X be a variety and let ν be a valuation on X. We say that x is the* ***centre*** *of ν on X if $\mathcal{O}_{X,x}$ is dominated by the valuation ring of ν.*

Lemma 8.10. *Let R be a local ring which is an integral domain with field of fractions K. Then R is a valuation ring if and only if it is maximal with respect to dominance. Every local ring in K is contained a valuation ring.*

The centre, if it exists, is unique. In the example above, the unique centre is x. It is easy to see, however, that the centre does not determine the valuation.

Example 8.11. *Let S be a smooth surface. Let $T \longrightarrow S$ be any sequence of blow ups with centre x. Let E be any exceptional divisor with generic point $t \in T$. Then t determines a valuation ν on T, whence on S. The centre of ν is x.*

Note however that if the centre of a discrete valuation is a divisor, then there is essentially only one way to define the valuation, as the order of vanishing. Given this, the next natural question to ask, is if it is true that every discrete valuation is associated to a divisor.

Definition 8.12. *A discrete valuation ν of X is called* ***algebraic*** *if there is a birational model Y of X such that the centre of ν on Y is a divisor.*

Example 8.13. *Consider the affine plane over $\mathbb{C}$. Considering only the closed points, the curve $y = e^x$, the graph of the exponential function, defines a discrete valuation of the local ring $\mathcal{O}_{S,p}$, where $S = \mathbb{A}^2_{\mathbb{C}}$ and p is the origin. Given $f \in \mathcal{O}_{S,p}$ we just consider to what order f approximates the curve above.*

Put differently the smooth curve $y = e^x$ determines an infinite sequence of blow ups with centre p. At each stage we blow up the unique point where the strict transform of $y = e^x$ meets the new exceptional divisor. The valuation ν then counts how many points the blow up of $f = 0$ shares.

It is clear that ν is not an algebraic valuation.

Definition 8.14. *Let X be a topological space. We say that x_0 is a* ***specialisation*** *of x_1 if $x_0 \in \overline{\{x_1\}}$.*

Lemma 8.15. *Let R be a valuation ring with quotient field K. Let $T = \operatorname{Spec} R$ and let $U = \operatorname{Spec} K$. Let X be any scheme.*

(1) To give a morphism $U \longrightarrow X$ is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subset K$.
(2) To give a morphism $T \longrightarrow X$ is equivalent to giving two points x_0, $x_1 \in X$, with x_0 a specialisation of x_1 and an inclusion of fields $k(x_1) \subset K$, such that R dominates the local ring $\mathcal{O}_{Z,x_0}$, in the subscheme $Z = \overline{\{x_1\}}$ of X, with its reduced induced structure.

Proof. We have already seen (1). Let t_0 be the closed point of T and let t_1 be the generic point. If we are given a morphism $T \longrightarrow X$, then let x_i be the image of t_i. As T is reduced, we have a factorisation $T \longrightarrow Z$. Moreover $k(x_1)$ is the function field of Z, so that there is a morphism of local rings $\mathcal{O}_{Z,x_0} \longrightarrow R$ compatible with the inclusion $k(x_1) \subset K$. Thus R dominates $\mathcal{O}_{Z,x_0}$.

Conversely suppose given x_0 and x_1. The inclusion $\mathcal{O}_{Z,x_0} \longrightarrow R$ gives a morphism $T \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0}$, and composing this with the natural map $\operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow X$ gives a morphism $T \longrightarrow X$. □

Lemma 8.16. *Let $f : X \longrightarrow Y$ be a compact morphism of schemes (that is, for every open affine subsheme $U \subset Y$, $f^{-1}(U)$ is compact).*

Then $f(X)$ is closed if and only if it is stable under specialisation.

Proof. Let us in addition suppose that f is of finite type and that X and Y are noetherian. Then $f(X)$ is constructible by Chevalley's Theorem, whence closed. For the general case, see Hartshorne, II.4.5. □

Now we are ready to state:

Theorem 8.17 (Valuative Criterion of Separatedness)**.** *Let $f : X \longrightarrow Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if the following condition holds:*

For any field K and for any valuation ring R with quotient field K, let $T = \operatorname{Spec} R$, let $U = \operatorname{Spec} K$ and let $i : U \longrightarrow T$ be the morphism induced by the inclusion $R \subset K$. Given morphisms $T \longrightarrow Y$ and $U \longrightarrow X$ which makes a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y, \end{array}$$

there is at most one morphism $T \longrightarrow X$ which makes the diagram commute.

Proof. Suppose that f is separated, and that we are given two morphisms $h : T \longrightarrow X$ and $h' : T \longrightarrow X$, which make the diagram commute.

Then we get a morphism $h'': T \longrightarrow X \underset{Y}{\times} X$. As the restrictions of h and h' to U agree, it follows that h'' sends the generic point t_1 of T to a point of the diagonal $\Delta(X)$. Since the diagonal is closed, it follows that t_0 is sent to a point of the diagonal. But then the images of t_0 and t_1, under h and h', are the same points x_0 and $x_1 \in X$. Since the inclusion $k(x_1) \subset K$ comes out the same, it follows that $h = h'$.

Now let us prove the other direction. It suffices to prove that $\Delta(X)$ is closed in $X \underset{Y}{\times} X$, which in turn is equivalent to proving that it is stable under specialisation. Suppose that $\xi_1 \in \Delta(X)$ and suppose that ξ_0 is in the closure of $\{\xi_1\}$. Let $K = k(\xi_1)$ and let A be the local ring of ξ_0 in the closure of ξ_1. Then $A \subset K$ and so there is a valuation ring R which dominates A. Then by (8.15) there is a morphism $T \longrightarrow X \underset{Y}{\times} X$, where $T = \operatorname{Spec} R$, sending t_i to ξ_i. Composing with either projection down to X, we get two morphisms $T \longrightarrow X$, which give the same morphism to Y and whose restrictions to U are the same, as $\xi_1 \in \Delta(X)$. By assumption then, these two morphisms agree, and so the morphism $T \longrightarrow X \underset{Y}{\times} X$ must factor through Δ. But then $\xi_0 \in \Delta(X)$, whence $\Delta(X)$ is closed. □

Corollary 8.18. *Assume that all schemes are Noetherian*

(1) Open and closed immersions are separated.
(2) A composition of separated morphisms is separated.
(3) Separated morphisms are stable under base change.
(4) If $f: X \longrightarrow Y$ and $f': X' \longrightarrow Y'$ are two separated morphisms over a scheme S, then the product morphism $f \times f': X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also separated.
(5) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is separated, then f is separated.
(6) A morphism $f: X \longrightarrow Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is separated for each i.

Proof. These all follow from (8.17). For example, consider the proof of (2). We are given $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$, two separated morphisms. By assumption we are given two morphisms $h: T \longrightarrow X$ and $h': T \longrightarrow X$, as in (8.17). By composition with f, these give two morphisms $k: T \longrightarrow Y$ and $k': T \longrightarrow Y$. As g is separated, these morphisms agree. But then as f is separated, $h = h'$. □

Now we turn to the notion of properness.

Definition 8.19. *A morphism $f\colon X \longrightarrow Y$ is **proper** if it is separated, of finite type, and universally closed.*

Example 8.20. *The affine line $\mathbb{A}^1_k$ is certainly separated and of finite type over k. However it is not proper, since it is not universally closed. Indeed consider $\mathbb{A}^2_k = \mathbb{A}^1_k \underset{k}{\times} \mathbb{A}^1_k$. The image of the hyperbola under projection down to $\mathbb{A}^1_k$ is not closed.*

Theorem 8.21 (Valuative Criterion of Properness)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of finite type, with X Noetherian. Then f is proper if and only if for every valuation ring R and for every pair of morphisms $U \longrightarrow Y$ and $T \longrightarrow Y$ which form a commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y, \end{array}$$

there is a unique morphism $h\colon T \longrightarrow X$ making the diagram commute.

Proof. Suppose that f is proper. Then f is certainly separated, so h, if it exists, is surely unique. Consider the base change given by $T \longrightarrow Y$, and set $X_T = X \underset{Y}{\times} T$. We get a morphism $U \longrightarrow X_T$, applying the universal property to the morphisms $U \longrightarrow X$ and $U \longrightarrow T$.

$$\begin{array}{ccccc} U & \longrightarrow & X_T & \longrightarrow & X \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ & & T & \longrightarrow & Y. \end{array}$$

Let $\xi_1 \in X_T$ be the image of the point $t_1 \in U$. Let $Z = \overline{\{\xi_1\}}$. As f is proper, f' is closed and so $f'(Z) \subset T$ is closed. Thus $f'(Z) = T$, as $f'(Z)$ contains the generic point of T. Pick $\xi_0 \in Z$ such that $f(\xi_0) = t_0$. Then we get a morphism of local rings $R \longrightarrow \mathcal{O}_{Z,\xi_0}$. The function field of Z is $k(\xi_1)$ which by construction is a subfield of K. On the other hand, R is maximal with respect to dominance in K. Thus $R \simeq \mathcal{O}_{Z,\xi_0}$. Hence by (8.15) there is a morphism $T \longrightarrow X_T$ sending t_i to ξ_i. Now compose with the natural map $X_T \longrightarrow X$.

Now suppose that f satisfies the given condition. Let $Y' \longrightarrow Y$ be an arbitrary base change, and let $X' \longrightarrow X$ be the induced morphism. Pick a closed subset $Z \subset X'$, imbued with the reduced induced structure:

$$\begin{array}{ccccc} Z & \subset & X' & \longrightarrow & X \\ & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ & & Y' & \longrightarrow & Y. \end{array}$$

We want to prove that $f(Z)$ is closed. f is of finite type by assumption, so that f' is of finite type. It suffices to show that $f(Z)$ is closed under specialisation, by (8.16).

Pick a point $z_1 \in Z$ and let $y_1 = f(z_1)$. Suppose that y_0 is a specialisation of y_1. Let S be the local ring of the closure of y_1 at y_0. Then the quotient field of S is $k(y_1)$ which is a subfield of $K = k(z_1)$. Pick a valuation ring R contained in K which dominates S. Then by (8.15), we get a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow \\ T & \longrightarrow & Y'. \end{array}$$

Composing with the morphisms $Z \longrightarrow X' \longrightarrow X$ and $Y' \longrightarrow Y$ we get morphisms $U \longrightarrow X$ and $T \longrightarrow Y$. By hypothesis, there is a morphism $T \longrightarrow X$ which makes the diagram commute. By the universal property of a fibre product, this lifts to a morphism $T \longrightarrow X$. As Z is closed, this factors into $T \longrightarrow Z$. Let z_0 be the image of t_0. Then z_0 maps to y_0, as required. □

Corollary 8.22. *Assume that all schemes are Noetherian*

(1) A closed immersion is proper.
(2) Composition of proper morphisms is proper.
(3) Proper morphisms are stable under base change.
(4) If $f\colon X \longrightarrow Y$ and $f'\colon X' \longrightarrow Y'$ are two proper morphisms over a scheme S, then the product morphism $f \times f'\colon X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also proper.
(5) If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is proper and g is separated, then f is proper.
(6) A morphism $f\colon X \longrightarrow Y$ is proper iff Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for each i.

Proof. These all follow from (8.21). □