6. Grasmannians

We first treat Grassmanians classically. Fix an algebraically closed field K. We want to parametrise the space of k-planes W in a vector space V. The obvious way to parametrise k-planes is to pick a basis $v_1, v_2, \dots, v_k$ for W. Unfortunately this does not specify W uniquely, as the same vector space has many different bases. However, the line spanned by the vector

$$\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^k V,$$

is invariant under re-choosing a basis.

Definition 6.1. *The* ***Grassmannian*** $G(k,V)$ ***of*** k***-planes in*** V *is the set of rank one vectors in* $\mathbb{P}(\bigwedge^k V)$.

We set $G(k,n) = G(k,K^n)$ and $\mathbb{G}(k,n) = G(k+1,n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$.

The embedding of the Grassmannian inside $\mathbb{P}(\bigwedge^k V)$ is known as the Plücker embedding. If we choose a basis $e_1, e_2, \dots, e_n$ for V, then a general element of $\bigwedge^k V$ is given by

$$\sum_I p_I e_I,$$

where I ranges over all collections of increasing sequences of integers between 1 and n,

$$i_1 < i_2 < \cdots < i_k,$$

and e_I is shorthand for the wedge of the corresponding vectors,

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

The coefficients p_I are naturally coordinates on $\mathbb{P}(\bigwedge^k V)$, which are known as the Plücker coordinates.

There is another way to look at the construction of the Grassmannian which is very instructive. If we pick a basis $e_1, e_2, \dots, e_n$ for V, then let A be the $k \times n$ matrix whose rows are $v_1, v_2, \dots, v_k$, in this basis. As before, this matrix does not uniquely specify $W \subset V$, since we could pick a new basis for W. However the operation of picking a new basis corresponds to taking linear combinations of the rows of our matrix, which in turn is the same as multiplying our matrix by a $k \times k$ invertible matrix on the left. In other words the Grassmannian is the set of equivalence classes of $k \times n$ matrices under the action of $\mathrm{GL}_k(K)$ by multiplication on the left.

It is not hard to connect the two constructions. Given the matrix A, then form all possible $k \times k$ determinants. Any such determinant is determined by specifying the columns to pick, which we indicate by a multindex I. In terms of $\bigwedge^k V$, this is the same as picking a basis and expanding our vector as a sum

$$\sum_I p_I e_I,$$

where, as before, e_I is the wedge of the corresponding vectors. For example consider the case $k = 2$, $n = 4$ (lines in $\mathbb{P}^3$). We have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

The corresponding plane is given as the span of the rows. We can form six two by two determinants. Clearly these are invariant, up to scalars, under the action of $\mathrm{GL}_2(K)$.

The Grassmannian has a natural cover by open affine subsets, isomorphic to affine space, in much the same way that projective space has a cover by open affines, isomorphic to affine space. Pick a linear space U of dimension $n - k$, and consider the set of linear spaces W of dimension k which are complementary to U, that is, which meet U only at the origin. Identify V with the sum $V/U + U$. Then a linear space W complementary to U can be identified with the graph of a linear map

$$V/U \longrightarrow U.$$

It follows that the subset of all linear spaces W complementary to U is equal to

$$\mathrm{Hom}(V/U, U) \simeq K^{k(n-k)} \simeq \mathbb{A}^{k(n-k)}_K.$$

Another way to see this is as follows. Consider the first $k \times k$ minor. Suppose that the corresponding determinant is non-zero, that is, the corresponding vectors are independent. In this case the $k \times k$ minor is equivalent to the identity matrix, and the only element of $\mathrm{GL}_k(K)$ which fixes the identity, is the identity itself. Thus we have a canonical representative of the matrix A for the linear space W. We are free to choose the other $k \times (n - k)$ block of the matrix, which gives us an affine space of dimension $k(n - k)$. The condition that the first $k \times k$ minor has non-zero determinant is an open condition, and this gives us an open affine cover by affine spaces of dimension $k(n - k)$. Note that the condition that the first $k \times k$ minor is invertible is equivalent to the condition that we do not meet the space given by the vanishing of the first k coordinates, which is indeed a linear space of dimension $n - k$.

It is interesting to write down the equations cutting out the image of the Grassmannian under the Plücker embedding, although this turns out to involve some non-trivial multilinear algebra. The problem is to characterise the set of rank one vectors ω in $\bigwedge^k V$.

Definition 6.2. *Let $\omega \in \bigwedge^k V$. We say that ω is **divisible** by $v \in V$ if there is an element $\phi \in \bigwedge^k V$ such that $\omega = \phi \wedge v$.*

Lemma 6.3. *Let $\omega \in \bigwedge^k V$.*

Then ω is divisible by v if and only if $\omega \wedge v = 0$.

Proof. This is easy. If $\omega = \phi \wedge v$, then

$$\begin{aligned} \omega \wedge v &= \phi \wedge v \wedge v \\ &= 0. \end{aligned}$$

To see the other direction, extend v to a basis $v = e_1, e_2, \dots, e_n$ of V. Then we may expand ω in this basis.

$$\omega = \sum p_I e_I.$$

On the other hand

$$v \wedge e_I = \begin{cases} e_J & \text{if } 1 \notin I, \text{ where } J = \{1\} \cup I \\ 0 & \text{if } 1 \in I. \end{cases}$$

Thus $\omega \wedge v = 0$ if and only if $p_I \neq 0$ implies $1 \in I$ if and only if v divides ω. □

Lemma 6.4. *Let $\omega \in \bigwedge^k V$.*

Then ω has rank one if and only if the linear map

$$\phi(\omega) \colon V \longrightarrow \bigwedge^{k+1} V \qquad\qquad v \longrightarrow \omega \wedge v,$$

has rank at most $n - k$.

Proof. Indeed $\phi(\omega)$ has rank at most $n - k$ if and only if the linear subspace of vectors dividing ω has dimension at least k if and only if ω has rank one. □

Now the map

$$\phi \colon \bigwedge^k V \longrightarrow \operatorname{Hom}(V, \bigwedge^{k+1} V),$$

is clearly linear. Thus the map ϕ can be interpreted as a matrix whose entries are linear coordinates of $\bigwedge^k V$ and the locus we want is given by the vanishing of the $(n - k + 1) \times (n - k + 1)$ minors.

It follows that the Grassmannian is a closed subset of $\mathbb{P}(\bigwedge^k V)$. Unfortunately the equations we get in this way won't be best possible. In particular they won't generate the ideal of the Grassmannian (they only cut out the Grassmannian set theoretically). To find equations that generate the ideal, we have to work quite a bit harder.

Lemma 6.5. *There is a natural pairing between* $\bigwedge^k V$ *and* $\bigwedge^{n-k} V^*$*. This pairing is well-defined up to scalars and preserves the rank.*

Proof. There is a natural pairing

$$\bigwedge^k V \times \bigwedge^{n-k} V \longrightarrow \bigwedge^n V,$$

which sends

$$(\omega, \eta) \longrightarrow \omega \wedge \eta.$$

On the other hand, $\bigwedge^n V$ is one dimensional so that it is non-canonically isomorphic to K and $(\bigwedge^{n-k} V)^*$ is isomorphic to $\bigwedge^{n-k} V^*$. □

Given ω, let ω^* be the corresponding element of $\bigwedge^{n-k} V^*$. Now there is a natural map

$$\psi(\omega^*)\colon V^* \longrightarrow \bigwedge^{n-k+1} V^*$$

which sends

$$v^* \longrightarrow \omega^* \wedge v^*.$$

Further ω has rank one if and only if ω^* has rank one, which occurs if and only if $\psi(\omega^*)$ has rank at most k.

Moreover the kernel of $\phi(\omega)$, namely W, is precisely the annihilator of the kernel of $\psi(\omega^*)$. Dualising, we get maps

$$\phi^*(\omega)\colon \bigwedge^{k+1} V^* \longrightarrow V^* \qquad \text{and} \qquad \psi^*(\omega)\colon \bigwedge^{n-k+1} V \longrightarrow V,$$

whose images annihilate each other if and only if ω has rank one.

Thus ω has rank one if and only if for every $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V^*$,

$$\Xi_{\alpha,\beta}(\omega) = \langle \phi^*(\omega)(\alpha), \psi^*(\omega)(\beta) \rangle = 0.$$

Now $\Xi_{\alpha,\beta}$ are quadratic polynomials, which are known as the Plücker relations. It turns out that they do indeed generate the ideal of the Grassmannian.

It is interesting to see what happens when $k = 2$:

Lemma 6.6. *Let* $\omega \in \bigwedge^2 V$*.*

Then ω *has rank one if and only if* $\omega \wedge \omega = 0$*.*

Proof. One direction is clear, in fact for every k, if ω has rank one then $\omega \wedge \omega = 0$.

To see the other direction, we need to prove that if ω has rank at least two, then $\omega \wedge \omega \neq 0$. First observe that if ω has rank at least two, then we may find a projection down to a vector space of dimension four, such that the image has rank two. Thus we may assume that V has dimension four and ω has rank two. In this case, up to change of coordinates,

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

and by direct computation, $\omega \wedge \omega$ is not zero. $\square$

Now

$$\omega = \sum_{i,j} p_{i,j} e_i \wedge e_j.$$

Suppose that $n = 4$. If we expand

$$\omega \wedge \omega,$$

then everything is a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We need to pick a term from each bracket, so that the union is $\{1, 2, 3, 4\}$. In other words, the coefficient of the expansion is a sum over all partitions of $\{1, 2, 3, 4\}$ into two equal parts. By direct computation, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

In particular, $\mathbb{G}(1, 3)$ is a quadric in $\mathbb{P}^5$ of maximal rank.