# 4. Images of Varieties

Given a morphism $f\colon X \longrightarrow Y$ of quasi-projective varieties, a basic question might be to ask what is the image of a closed subset $Z \subset X$. Replacing $X$ by $Z$ we might as well assume that $Z = X$.

At first this question seems quite hopeless; indeed our first hope is that the image of $X$ is always a quasi-projective subvariety. Unfortunately this is definitely not true. For example, take $X = Y = \mathbb{A}^2$. Let $\pi\colon X \longrightarrow Y$ be the morphism $(a, b) \longrightarrow (a, ab)$. Let us determine the image. Pick $(x, y) \in \mathbb{A}^2$. If $x \neq 0$, then take $a = x$ and $b = y/x$. Then $(a, ab) = (x, y)$. Thus the image contains the complement of the $x$-axis. Now if $y \neq 0$ and $x = 0$, then $(x, y)$ is surely not in the image. However $(0, 0)$ is in the image; indeed it is the image of $(0, 0)$. Thus the image is equal to the complement of the $x$-axis union the origin.

In fact, it turns out that this is as bad as it gets. The first case to deal with, in fact the crucial case, which is of interest in its own right, is the case when $X$ is projective.

**Definition 4.1.** *Let $f\colon X \longrightarrow Y$ be a function between two topological spaces. We say that $f$ is* ***proper*** *if $f$ takes closed sets to closed sets.*

**Theorem 4.2.** *Every morphism $\pi\colon X \longrightarrow Y$ of varieties, where $X$ is projective, is proper.*

**Definition 4.3.** *Let $i\colon X \longrightarrow Y$ be a morphism. We say that $i$ is* ***closed*** *if the image of $X$ is closed. We say that $i$ is a* ***closed embedding*** *if $i$ is closed and $i$ is an isomorphism onto its image.*

**Definition 4.4.** *Let $\pi\colon X \longrightarrow Y$ be a morphism.*

*We say that $\pi$ is a* ***projective morphism*** *if it can be factored into a closed embedding $i\colon X \longrightarrow Y \times \mathbb{P}^n$ and the projection morphism $Y \times \mathbb{P}^n \longrightarrow Y$.*

Obvious examples of projective morphisms are blow ups. Also

**Lemma 4.5.** *Every morphism from a projective variety is projective.*

*Proof.* Just take the graph. $\square$

Clearly closed embeddings are proper and the composition of proper maps is proper. Thus to prove (4.2) it suffices to prove:

**Theorem 4.6.** *Every projective morphism is proper.*

Moreover we may assume that $X \subset Y \times \mathbb{P}^n$ and that we are projecting onto the first factor. The trick is to reduce to the case $n = 1$. The idea is that projective space $\mathbb{P}^n$, via projection, is very close to the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

**Definition 4.7.** *Let $\pi \colon X \longrightarrow Y$ be a morphism of varieties. We say that $\pi$ is a fibre bundle, with fibre $F$, if we can find a cover of the base $Y$, such that over each open subset $U$ of the cover, $\pi^{-1}(U) \simeq U \times F$.*

Note that if $\pi$ is a fibre bundle then every fibre of $\pi$ is surely a copy of $F$. It is convenient to denote $\pi^{-1}(U)$ by $X|_U$.

**Lemma 4.8.** *The graph of the projection map from a point $p$ defines a morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$, which is a fibre bundle, with fibre $\mathbb{P}^1$.*

*Proof.* The rational map given by projection from a point $p$

$$\pi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

is clearly defined everywhere, except at the point of projection. Moreover this map is clearly constant on any line through $p$. Thus the morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$ has fibres equal to the lines trough $p$.

Pick two hyperplanes $H_1$ and $H_2$, neither of which contain $p$. Under projection, we may indentify $H_1$ with the base $\mathbb{P}^{n-1}$. Let $V = H_1 \cap H_2$. Then the image of $V$ is a hyperplane in $\mathbb{P}^{n-1}$. Let $U$ be the complement. Projection from $V$ defines a rational map down to $\mathbb{P}^1$. This rational map is an isomorphism on every line $l$ through $p$ which does not intersect $V$.

Define a morphism $\psi \colon \Gamma|_U \longrightarrow \mathbb{P}^1 \times U \subset \mathbb{P}^1 \times \mathbb{P}^{n-1}$ via these two projection maps. It is not hard to see that $\psi$ is an isomorphism. Fixing $H_1$ and varying $H_2$ it is clear that we get a cover of $\mathbb{P}^{n-1}$ in this way. ☐

One nice property of both (4.2) and (4.6) is that they may be checked locally on the base.

**Lemma 4.9.** *Let $U_\alpha$ be an open cover of $Y$.*

*Then $\pi(X)$ is closed if and only if $\pi(X|_{U_\alpha})$ is closed, for every $\alpha$.*

*Proof.* Clear, since a subset $A \subset Y$ is closed if and only if its intersection with every element of the open cover is closed. ☐

**Lemma 4.10.** *To prove (4.6) we may assume that $n = 1$.*

*Proof.* Let $X \subset Y \times \mathbb{P}^n$. If $X = Y \times \mathbb{P}^n$ then there is nothing to prove, since the image is the whole of $Y$. Otherwise pick a point $p$ such that $X$ is not contained in $Y \times \{p\}$. Let $q \colon Y \times \Gamma \longrightarrow Y \times \mathbb{P}^n$ be the blow up of $Y \times \{p\}$ (equivalently blow up $\mathbb{P}^n$ at $p$ and then cross with $Y$). Let $X'$ be the strict transform of $X$. Then the image of $X'$ and $X$ in $Y$ coincide.

Now by (4.8), the morphism $Y \times \Gamma \longrightarrow Y$ factors through $Y \times \mathbb{P}^{n-1}$. By induction on $n$, it suffices to prove that the image of $X \subset Y \times \Gamma$ inside in $Y \times \mathbb{P}^{n-1}$ is closed.

By (4.9) we are free to replace $Y \times \mathbb{P}^{n-1}$ by any open subset. Then by (4.8) we may assume $\Gamma = \mathbb{P}^{n-1} \times \mathbb{P}^1$. Replacing $Y$ by $Y \times \mathbb{P}^{n-1}$ we are done. $\square$

The idea now is to work locally on $Y$ and think of $Y \times \mathbb{P}^1$ as being $\mathbb{P}^1$ over a funny field.

**Lemma 4.11.** *Let $X \subset Y \times \mathbb{P}^1$ be a closed subset.*

*Then locally about every point of $Y$, $X$ is defined by polynomials $F(S,T)$, where $[S : T]$ are homogeneous coordinates on $\mathbb{P}^1$ and the coefficients of $F$ belong to the coordinate ring of $Y$.*

*Proof.* This is easy. If $Y$ is affine, then we can cover $Y \times \mathbb{P}^1$ by two open affine sets $Y \times U_0$ and $Y \times U_1$. In this case $X$ is locally defined, on each piece, by polynomials $f(s)$ and $g(t)$, where $s = S/T$ and $t = T/S$ and the coefficients of $f$ and $g$ belong to $A(Y)$. Since $f(s) = g(t)$ on $Y \times (U_0 \cap U_1)$ it follows that there is a global polynomial $F(S,T)$ with coefficients in $A(Y)$ which on each piece affine piece reduces to $f(s)$ and $g(t)$. $\square$

In other words, we only need to consider polynomials $F(S,T) \in A(Y)[S,T]$. Given $y \in Y$, let $F_y = F_y(S,T) \in K[S,T]$ be the polynomial we obtain by subsituting in $y \in Y$ to the coefficients.

**Lemma 4.12.** *Let $X \subset Y \times \mathbb{P}^1$.*

*Then $y \in \pi(X)$ if and only if for every pair of functions $F(S,T)$ and $G(S,T) \in A(Y)[S,T]$ vanishing on $X$, both $F_y(S,T)$ and $G_y(S,T)$ have a common zero on $\{y\} \times \mathbb{P}^1$.*

*Proof.* One inclusion is clear. So suppose that $y \notin \pi(X)$. Pick $F(S,T)$ that does not vanish on $\{y\} \times \mathbb{P}^1$. Then $F_y(S,T)$ has only finitely many zeroes. For each such zero $p_i$, we may find $G^i(S,T)$ such that $G^i_y(S,T)$ does not vanish at $p_i$. Taking an appropriate linear combination of the $G^i$ gives us a polynomial $G$ such that $F_y$ and $G_y$ do not have a common zero. $\square$

**Lemma 4.13.** *To prove* (4.6) *we may assume that $X$ is defined by two polynomials $F$ and $G$.*

To finish off, the idea is to use elimination theory.

**Definition-Lemma 4.14.** *Let $A$ be a ring, and let $F$ and $G$ be two polynomials in $A[S,T]$, of degrees $d$ and $e$.*

*Let $R(F,G) \in A$ be the determinant of the following matrix*

$$\begin{vmatrix} f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots & \dots \\ 0 & f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & f_0 & f_1 & f_2 & \dots & f_d \\ g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots & \dots \\ 0 & g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & g_0 & g_1 & g_2 & \dots & g_e \end{vmatrix},$$

*where $f_0, f_1, \dots, f_d$ and $g_0, g_1, \dots, g_e$ are the coefficients of $F$ and $G$.*

*Then for every maximal ideal $\mathfrak{m}$ of $A$, $\bar{R}(F,G) = 0$ in the quotient ring $S/\mathfrak{m}$ if and only if the two polynomials $\bar{f}$ and $\bar{g}$ have a common zero.*

*Proof.* Since expanding a determinant commutes with passing to the quotient $A/\mathfrak{m}$, we might as well assume that $S = K$ is a field.

Now note that the rows of this matrix correspond to the polynomials $S^iT^{e-1-i}F$ and $S^jT^{d-1-j}G$, where $0 \leq i \leq e-1$ and $0 \leq j \leq d-1$, expanded in the standard basis of the vector space $P_{d+e-1}$ of polynomials of degree $d+e-1$. Thus the determinant is zero if and only if the polynomials $B = \{S^iT^{e-1-i}F, S^jT^{d-1-j}G\}$ are dependent, inside $P_{d+e-1}$.

To finish off then it suffices to prove that this happens only when the two polynomials share a common zero. Now note that $P_{d+e-1}$ has dimension $d+e$. Thus the $d+e$ polynomials $B$ are independent if and only if they are a basis. Suppose that they share a common zero. Then the space spanned by $B$ is contained in the vector subspace of all polynomials vanishing at the given point, and so $B$ does not span. Now suppose that they are dependent. Collecting terms, there are then two polynomials $P$ and $Q$ of degrees $e-1$ and $d-1$ such that

$$PF + QG = 0.$$

Suppose that $d \leq e$. Every zero of $G$ must be a zero of $PF$. As $G$ has $e$ zeroes and $P$ has at most $e-1$ zeroes, it follows that one zero of $G$ must be a zero of $F$. $\square$

*Proof of* (4.6). By (4.13) it suffices to prove the result when $n = 1$ and $X$ is defined by two polynomials $F$ and $G$. In this case $\pi(X)$ is precisely given by the resultant of $F$ and $G$, which is an element of $A(Y)$. $\square$

(4.2) has the following very striking consequence.

**Corollary 4.15.** *Every regular function on a connected projective variety is constant.*

*Proof.* By definition a regular function is a morphism $f\colon X \longrightarrow \mathbb{A}^1$. Now by (4.2) the image of $X$ is closed in $\mathbb{A}^1$. The only closed subsets of $\mathbb{A}^1$ are finite sets of points or the whole of $\mathbb{A}^1$. On the other hand $f$ extends in an obvious way to a morphism $g\colon X \longrightarrow \mathbb{P}^1$. We haven't changed the image, but the image is now also a closed subset of $\mathbb{P}^1$. Thus the image cannot be $\mathbb{A}^1$.

Thus the image is a finite set of points. As $X$ is connected, the image is connected and so the image is a point. □

**Corollary 4.16.** *Let $X$ be a closed and connected subset of an affine variety.*

*If $X$ is also projective then $X$ is a point.*

*Proof.* By assumption $X \subset \mathbb{A}^n$. Suppose that $X$ contains at least two points. Then at least one coordinate must be different. Let $f$ be the function on $\mathbb{A}^n$ corresponding to this coordinate. Then $f$ restricts to a non-constant regular function on $X$, which contradicts (4.15). □

**Corollary 4.17.** *Let $X \subset \mathbb{P}^n$ be a closed subset and let $H$ be a hypersurface.*

*If $X$ is not a finite set of points, then $H \cap X$ is non-empty.*

*Proof.* Suppose not. Let $G$ be the defining equation of $H$. Pick $F$ of degree equal to the degree of $G$. Then $F/G$ is a regular function on $X$, since $G$ is nowhere zero on $X$. But this contradicts (4.15). □

We can now answer our original question.

**Definition 4.18.** *Let $X$ be a topological space. A subset $Z \subset X$ is said to be **constructible** if it is the finite union of locally closed subsets.*

Note that constructible sets are closed under complements and finite intersections and unions.

**Lemma 4.19.** *Let $X$ be a Noetherian topological space and let $Z$ be a subset.*

*Then $Z$ is constructible if and only if it is of the form*

$$Z = Z_0 - (Z_1 - (Z_2 - \cdots - Z_k)),$$

*where $Z_i$ are closed and decreasing subsets.*

*Proof.* Suppose that $Z$ is constructible. Let $Z_0$ be the closure of $Z$. Then $Z$ is dense in $Z_0$ and $Z_0$ is closed. As $Z$ is constructible, it contains a dense open subset of $Z_0$. Clearly the difference $Z_0 - Z$ is constructible. Let $Z_1$ be the closure. Then $Z_1$ is a proper closed subset of $Z_0$. Continuing in this way, we construct a decreasing sequence of closed subsets,

$$Z_0 \supset Z_1 \supset \cdots \supset Z_k \supset \ldots$$

As $X$ is Noetherian this sequence must terminate.

Now suppose that $Z$ is an alternating difference of closed subsets,

$$Z = Z_0 - (Z_1 - (Z_2 - \cdots - Z_{2k+1})).$$

Then

$$Z = (Z_0 - Z_1) \cup (Z_2 - Z_3) \cup \cdots \cup (Z_{2k} - Z_{2k+1}).$$ □

**Theorem 4.20** (Chevalley's Theorem). *Let $\pi \colon X \longrightarrow Y$ be a morphism of quasi-projective varieties.*

*Then the image of a constructible set is constructible.*

*Proof.* As the image of a union is the union of the images, it suffices to prove that the image of a locally closed subset is constructible. Suppose that $Z$ is a locally closed subset. Replacing $X$ by the closure of $Z$ and $Y$ by the closure of the image, we may assume that $\pi|_Z$ is dominant. Suppose that $\pi(Z)$ contains an open subset. Replacing $X$ by the complement of the inverse image, we are then done by Noetherian induction.

Thus we are reduced to proving that $\pi(Z)$ contains an open subset. Replacing $X$ by an open subset, we may assume that $X$ is affine. Replacing $X$ by its graph and applying induction on $n$, we may assume that $Z \subset \mathbb{A}^n$ and that the map is the restriction of the projection map

$$\mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$$

where

$$(x_1, x_2, \ldots, x_n) \longrightarrow (x_1, x_2, \ldots, x_{n-1}).$$

Thus we may assume that $Z \subset Y \times \mathbb{A}^1$ and that we are projecting onto $Y$. Clearly we may replace $\mathbb{A}^1$ by $\mathbb{P}^1$. Working locally, we may assume that every closed subset of $Y \times \mathbb{P}^1$ is defined by polynomials of the form $F(S, T)$.

Let $X$ be the closure of $Z$ and let $V$ be the complement, so that $X = Z - V$ and both $Z$ and $V$ are closed. Suppose that $X = Y \times \mathbb{P}^1$. In this case it suffices to prove that

$$V_Y = \{\, y \in Y \mid \{y\} \times \mathbb{P}^1 \subset V \,\},$$

is contained in a proper closed subset. But $V$ is a proper closed subset so that there is a polynomial $G$ vanishing on $V$. In this case, $V$ contains the whole fibre if and only if every coefficient of $G_y$ vanishes. Thus the locus $V_Y$ is contained in the vanishing locus of all the coefficients of $G$.

So we may assume that $X$ is a proper closed subset of $Y \times \mathbb{P}^1$. Thus there is a polynomial $F$ vanishing on $X$. Since $X$ is closed, its image is closed, whence the whole of $Y$. It suffices to prove that $\pi(V)$ is a proper closed subset. It is certainly closed, as $V$ is closed. But $R(F, G)$ is a non-zero polynomial that vanishes on the image. $\square$