3. Rational Varieties

Definition 3.1. *A **rational function** is a rational map to $\mathbb{A}^1$.*

*The set of all rational functions, denoted $K(X)$, is called the **function field**.*

Lemma 3.2. *Let X be an irreducible variety.*

Then the function field is a field. If $U \subset X$ is any open affine subset, then the function field is precisely the field of fractions of the coordinate ring of U.

Proof. Clear, since on an irreducible variety, any rational function is determined by its restriction to any open subset, and locally any morphism is given by a rational function. □

Proposition 3.3. *Let K be an algebraically closed field.*

Then there is an equivalence of categories between the category of irreducible varieties over K with morphisms the dominant rational maps, and the category of finitely generated field extensions of K.

Proof. Define a functor F from the category of varieties to the category of fields as follows. Given a variety X, let $K(X)$ be the function field of X. Given a rational map $\phi\colon X \dashrightarrow Y$, define $F(\phi)\colon K(Y) \longrightarrow K(X)$ by composition. If f is a rational function on Y, then $\phi \circ f$ is a rational function on X.

We have to check that F is essentially surjective and fully faithful. Suppose that L is a finitely generated field extension of K. Then $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $A = K[\alpha_1, \alpha_2, \dots, \alpha_n]$. Let X be any affine variety with coordinate ring A. Then X is irreducible as A is an integral domain and the function field of X is precisely L as this is the field of fractions of A.

The fact that F is fully faithful is proved in the same way as before. □

Proposition 3.4. *Let X and Y be two irreducible varieties.*

Then the following are equivalent

1. *X and Y are birational.*
2. *X and Y contain isomorphic open subsets.*
3. *The function fields of X and Y are isomorphic.*

Proof. We have already seen that (1) and (3) are equivalent and clearly (2) implies (1) (or indeed (3)). It remains to prove that if X and Y are birational then they contain isomorphic open subsets.

Let $\phi\colon X \dashrightarrow Y$ be a birational map with inverse $\psi\colon Y \dashrightarrow X$. Suppose that ϕ is defined on U and ψ is defined on V. Then $\psi \circ \phi$ is defined on $\phi^{-1}(V)$ and it is equal to the identity there, since it is the identity on some dense open subset. Similarly $\phi \circ \psi$ is the identity on $\phi^{-1}(U)$. Then the open subset $\phi^{-1}(\psi^{-1}(U))$ of X and $\psi^{-1}(\phi^{-1}(V))$ of Y are isomorphic open subsets. $\square$

Corollary 3.5. *Let X be an irreducible variety.*
Then the following are equivalent
1. *X is rational.*
2. *X contains an open subset of $\mathbb{P}^n$.*
3. *The function field of X is a purely transcendental extension of K.*

Proof. Immediate from (3.4). $\square$

Let us consider some examples. I claim that the curve

$$C = V(y^2 - x^2 - x^3)$$

is rational. We have already seen that there is a morphism

$$\mathbb{A}^1 \longrightarrow C \qquad \text{given by} \qquad t \longrightarrow (t^2 - 1, t(t^2 - 1)).$$

We want to show that it is a birational map. One way to proceed is to construct the inverse. In fact the inverse map is

$$C \dashrightarrow \mathbb{A}^1 \qquad \text{given by} \qquad (x, y) \longrightarrow y/x.$$

Another way to proceed is to prove that the function field is purely transcendental. Now the coordinate ring is

$$K[x, y]/\langle y^2 - x^2 - x^3\rangle.$$

So the fraction field is $K(x, y)$, where $y^2 = x^2 + x^3$. Consider $t = y/x$. I claim that $K(t) = K(x, y)$. Clearly there is an inclusion one way. Now

$$t^2 = y^2/x^2 = 1 + x \qquad \text{and so} \qquad x = t^2 - 1 \in K(t).$$

But $y = tx$, so that we do indeed have equality $K(t) = K(x, y)$. Thus C is rational.

Perhaps a more interesting example is to consider the Segre variety $V \subset \mathbb{P}^3$. Consider projection π from a point p of the Segre variety,

$$\pi \colon V \dashrightarrow \mathbb{P}^2.$$

Clearly the only possible point of indeterminancy is the point p. Since a line, not contained in V, meets the Segre variety in at most two points, it follows that this map is one to one outside p, unless that line is contained in V. On the other hand, if $q \in \mathbb{P}^2$, the line $\langle p, q\rangle$ will meet the Segre variety in at least two points, one of which is p.

Now through the point p, there passes two lines l and m (one line of each ruling). These get mapped to two separate points, say q and r. It follows that p is indeed a point of indeterminancy. To proceed further, it is useful to introduce coordinates. Suppose that $p = [0 : 0 : 0 : 1]$, where $V = V(XW - YZ)$.

Now projection from $p \in \mathbb{P}^3$ defines a rational map

$$\phi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,$$

whose exceptional locus is a copy of $\mathbb{P}^2$. Indeed the graph of ϕ lies in $\mathbb{P}^3 \times \mathbb{P}^2$ and as before over the point p, we get a copy of the whole of the image $\mathbb{P}^2$, as can be seen by looking at lines through p. Working on the affine chart $W \neq 0$, V is locally defined as $x = yz$. If $[R : S : T]$ are coordinates on $\mathbb{P}^2$, the equations for the blow up of $\mathbb{P}^3$ are given as

$$xS = yR \qquad xT = zR \qquad yT = zS.$$

The blow up of V at p is given as the strict transform of V in the blow up of $\mathbb{P}^3$. We work in the patch $T \neq 0$. Then $x = rz$ and $y = sz$ so that the we get the equation

$$rz - sz^2 = z(r - sz) = 0.$$

Now $z = 0$ corresponds to the whole exceptional locus so that $r = sz$ defines the strict transform. In this case $z = 0$, means $r = 0$, so that we get a line in the exceptional $\mathbb{P}^2$.

In other words the graph of π is the blow up of p, with an exceptional divisor isomorphic to $\mathbb{P}^1$. The graph of π then blows down the strict transform of the two lines. Note that the image of the exceptional divisor, is precisely the line connecting the two points q and r.

To see that π is birational, we write down the inverse,

$$\psi \colon \mathbb{P}^2 \dashrightarrow V.$$

Given $[R : S : T]$, we send this to $[R : S : T : ST/R]$. Clearly this lies on the quadric $XW - YZ$ and is indeed the inverse map. Note that the inverse map blows up q and r then blows down the line connecting them to p.

In fact it turns out that the picture above for rational maps on surfaces is the complete picture.

Theorem 3.6 (Elimination of Indeterminancy). *Let $\phi \colon S \dashrightarrow Z$ be a rational map from a smooth surface.*

Then there is an iterated sequence of blow ups of points $p \colon T \longrightarrow S$ such that the induced rational map $\psi \colon T \longrightarrow Z$ is a morphism.

Theorem 3.7. *Let $\phi \colon S \dashrightarrow T$ be a birational map of smooth surfaces.*

Then there is an iterated sequence of blow ups of points $p\colon W \longrightarrow S$ such that the induced map $q\colon W \longrightarrow T$ is also an iterated sequence of blow up of points, composed with an isomorphism.

In fact it turns out that both of these results generalise to all dimensions. In the first result, one must allow blowing up the ideal of any smooth subvariety. In the second result, one must allow mixing up the sequence of blowing up and down, although it is conjectured that the one can perform first a sequence of blow ups and then a sequence of blow downs.

Another way to proceed, is to compute the field of fractions. The coordinate ring on the affine piece $W \neq 0$ is

$$K[x, y, z]/\langle x - yz\rangle = k[y, z].$$

The field of fractions is visibly then $K(y, z)$. However perhaps the easiest way to proceed is to observe that $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$, so that the Segre Variety is clearly rational.

In fact it turns out in general to be a vary hard problem to determine which varieties are rational. As an example of this consider Lüroth's problem.

Definition 3.8. *We say that a variety X is* ***unirational*** *if there is a dominant rational map $\phi\colon \mathbb{P}^n \dashrightarrow X$.*

Question 3.9 (Lüroth)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory.

In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions. The first step is in fact to establish (3.6) and (3.7).

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\frac{\mathbb{Z}[x]}{\langle x^4\rangle}$, and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H^3(X, \mathbb{Z})$. They then give a reasonably elementary construction of a threefold with non-zero torsion in H^3.

Another proof is due to Clemens and Griffiths. It is not hard to prove that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. To prove this consider the family of lines on the cubic. It turns out that this is a two dimensional family, and that a lot of the geometry of the cubic is controlled by the geometry of this surface.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.