16. Cubics II

It turns out that the question of which varieties are rational is one of the subtlest geometric problems one can ask. Since the problem of determining whether a variety is rational or not is so delicate, various intermediary notions have been introduced.

Definition 16.1. *We say that a variety X is **unirational** if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$.*

Theorem 16.2. *Every smooth cubic threefold $V \subset \mathbb{P}^4$ is unirational.*

Note some basic properties of unirational varieties.

Lemma 16.3. *Let X be a variety over k. The following are equivalent:*

1. *X is unirational.*
2. *There is a dominant generically finite morphism $\phi\colon Y \longrightarrow X$, where Y is rational.*
3. *The function field of X is contained in a purely transcendental field extension of k.*
4. *There is a finite extension of the function field of X which is a purely transcendental field extension of k.*

Proof. The fact that (1) and (3) are equivalent, follows from the equivalence of categories between dominant rational maps and inclusions of function fields, and (4) follows from (2) in a similar fashion.

So suppose that $\phi\colon \mathbb{P}^n_k \dashrightarrow X$ is a dominant rational map. Replacing $\mathbb{P}^n$ by the normalisation of the graph of ϕ, we may assume that there a quasi-projective variety Y and a dominant morphism $Y \longrightarrow X$. If the dimension of the generic fibre is greater than zero, then pick a hyperplane $H \subset \mathbb{P}^n$, whose inverse image in Y dominates X. Continuing in this way, we reduce to the case where is generically finite. □

Thus to prove (16.2) we are looking for a dominant rational map $\mathbb{P}^3 \dashrightarrow V$. The trick is to consider low degree rational curves on V.

Lemma 16.4. *Every smooth cubic threefold V be in $\mathbb{P}^4$ contains a two dimensional family F of lines.*

Proof. Consider the incidence correspondence

$$\Sigma = \{ (l, H) \,|\, l \subset S = H \cap V \} \subset \mathbb{G}(1, 3) \times \mathbb{P}^4.$$

This has two morphisms, $p\colon \Sigma \longrightarrow \mathbb{G}(1, 3)$ and $q\colon \Sigma \longrightarrow \mathbb{P}^3$. Let H be a general hyperplane in $\mathbb{P}^4$. Then $S = H \cap V$ is a smooth cubic surface in $\mathbb{P}^3$. But then we have already seen that S contains a finite number of lines. Thus the minimum dimension of the fibres of p is zero. It follows that Σ has dimension four.

If we fix l then there is a two dimensional family of hyperplanes containing l (in fact a copy of $\mathbb{P}^2$). Since the fibres of q are two dimensional, it follows that $F_1 = q(\Sigma)$ has dimension two. $\square$

It is interesting to observe that there is a four dimensional family of conics. If you fix a line l and look at the family of planes containing the line then this will cut the cubic in a plane cubic curve. Part of this curve is the line l, and the residual curve is a conic. The family of planes containing the line is a copy of $\mathbb{P}^2$ and so a four dimensional family of conics.

The idea to prove (16.2) is to exploit the family of conics residual to a line.

Definition 16.5. *A **conic bundle** is a projective morphism, $\pi\colon X \longrightarrow S$, between quasi-projective varieties, where the fibres are conics in $\mathbb{P}^2$. A **rational conic bundle**, is any morphism, which is a conic bundle over an open subset of the base.*

Of course the fibres of any conic bundle have three types

- a smooth conic,
- a pair of lines,
- a double line.

Note that a morphism is a rational conic bundle if and only if the generic fibre is a smooth conic in $\mathbb{P}^2_K$, where K is the function field of the base. We will change our conventions a little; for now on in this section a variety is a separated scheme of finite type over a field, not necessarily algebraically closed. If the groundfield is not algebraically closed, then this question can become very tricky, even in low dimensions.

Lemma 16.6. *Let $\pi\colon X \longrightarrow S$ be a morphism of quasi-projective varieties.*

If the generic fibre is rational and S is rational then X is rational.

Proof. By assumption the function field of S is a purely transcendental extension of the groundfield k, $K = K(S) \simeq k(x_1, x_2, \dots, x_m)$. Equivalently the residue field of the generic point η of S is purely transcendental over k. Let X_η be the generic fibre. The function field of X_η is a purely transcendental extension of K. This is the residue field of the generic point ξ of X_η, which is also the residue field of X.

Thus the function field of X is a purely transcendental extension of a purely transcendental extension, so that it is a purely transcendental extension of k. Thus X is rational. $\square$

Lemma 16.7. *Let $V \subset \mathbb{P}^4$ be a smooth cubic.*

Then the blow up of V along a line is a rational conic bundle over $\mathbb{P}^2$.

Definition 16.8. *A k-**rational point** of a scheme X over S is any point which is the image of a morphism $\operatorname{Spec} k \longrightarrow X$ over S. The set of all k-rational points is denoted $X(k)$.*

In other words a k-rational point is simply a point whose residue field is a subfield of k.

Example 16.9. *Let $X = \mathbb{A}^1_{\mathbb{R}}$. Then $p = \langle x^2 + a\rangle \in X$, where $a \in \mathbb{R}$ and $a > 0$, corresponds to two $\mathbb{C}$-valued points. Indeed, there are two scheme maps*

$$\operatorname{Spec} \mathbb{C} \longrightarrow X,$$

whose image is p, corresponding to the fact that there are two automorphisms of $\operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{R}$, given by the identity and complex conjugation.

Lemma 16.10. *Let $C \subset \mathbb{P}^2_k$ be a smooth conic, over a field k.*

Then $C \simeq \mathbb{P}^1_k$ if and only if C contains a k-rational point

Proof. One direction is clear as $\mathbb{P}^1_k$ certainly contains k-rational points.

Now suppose that C contains a k-rational point. After applying an element of $\mathrm{PGL}(3, k)$, we may assume that this point is $[0 : 0 : 1]$. Consider projection from this point. This defines a morphism $C - [0 : 0 : 1] \longrightarrow \mathbb{P}^1$, which is surely defined over k (indeed it is the restriction of $[x : y : z] \longrightarrow [x : y]$). It is then straightforward to check that this morphism extends to an isomorphism. □

Example 16.11. *The conic $C = V(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ is not rational over $\operatorname{Spec} \mathbb{R}$.*

Definition 16.12. *Let $\pi \colon X \longrightarrow S$ be a morphism of schemes. A **section** of π is a morphism $\sigma \colon S \longrightarrow X$ such that $\sigma \circ \pi$ is the identity. A **rational section** is a section defined on some open subset U of S.*

Lemma 16.13. *Let $\pi \colon X \longrightarrow S$ be a morphism of integral schemes, of finite type. Then picking a rational section of π is equivalent to picking a rational point of the generic fibre.*

Proof. Let K be the function field of S. We may as well assume that both $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, so that K is the field of fractions of A. The generic fibre has coordinate ring $B \underset{A}{\otimes} K$. Suppose that we have a rational section. Then we may as well assume that we have a section. But this is equivalent to a ring homomorphism $B \longrightarrow A$. In turn this induces a ring homomorphism $B \underset{A}{\otimes} K \longrightarrow K$

which is equivalent to a morphism $\operatorname{Spec} K \longrightarrow X_\xi$, where ξ is the generic point of S. But this is exactly the same as a rational point of the general fibre.

Now suppose that we have a rational point of the generic fibre. This is equivalent to a ring homorphism $B \underset{A}{\otimes} K \longrightarrow K$. Since we have a morphism of finite type, B is a finitely generated A-algebra. Pick generators $b_1, b_2, \dots, b_k$. Denote the image of b_i in K by c_i/d_i, where c_i and d_i are elements of A. Passing to the open affine subset U_d of S, where d is the product $d_1 \cdot d_2 \cdot \dots \cdot d_k$, we may assume that $d_i = 1$, so that we get a morphism $B \longrightarrow A$. But this is equivalent to a section of π. $\square$

Proposition 16.14. *Let $\pi \colon X \longrightarrow S$ be a rational conic bundle, between two varieties, over an algebraically closed field k. Let $T \subset X$ be a subvarety of X which dominates S.*

(1) If T is unirational, then so is X.
(2) If $T \longrightarrow S$ is birational and T is rational, then so is X.

Proof. Consider the base change $T \longrightarrow S$ of X. Let Y be a component of the base change of maximal dimension, which dominates X. Then $Y \longrightarrow T$ is a conic bundle. Moreover, there is a natural morphism $T \longrightarrow Y$ which is a section. Possibly base changing further, we may assume that the base is rational, and that there is a rational section. Thus it suffices to prove (2).

Consider the generic fibre. By assumption it is a smooth conic in $\mathbb{P}^2_K$, where K is not algebraically closed. The rational section implies that this conic has a rational point. But then this conic is isomorphic to $\mathbb{P}^1_K$. The function field of this conic is then $K(t)$. The generic point of X is also the ceneric point of the generic fibre. It follows that the function field of X is isomorphic to $K(t)$. Since K is purely transcendental over k the groundfield, this implies that the field of fractions of X is purely transendental over k. But then X is rational. $\square$

Proof of (16.2). Let V be a smooth cubic threefold and let l be a line in V and let X be the blow up of V along l, $f \colon X \longrightarrow V$. Then there is a conic bundle $\pi \colon X \longrightarrow \mathbb{P}^2$. Let E be the excpetional divisor of the blow up. Then E is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Thus E is rational. But E dominates $\mathbb{P}^2$ and we are done by (16.14). $\square$

In fact $E \longrightarrow \mathbb{P}^2$ is a two to one map.

Question 16.15 (Lüroth's problem)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory. There is also a simple geometric proof of this fact (essentially the Riemann-Hurwitz formula). In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions.

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\mathbb{Z}[x]/\langle x^3\rangle$ and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H_3(X,\mathbb{Z})$. They then give a reasonably elementary construction of a threefold with non-zero torsion in H_3.

Another proof is due to Clemens and Griffiths. (16.2) implies that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. A lot of the geometry of the cubic is controlled by the geometry of the Fano surface of lines.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.

Since it is so hard to distinguish between rational and unirational, yet another closely related notion has been introduced.

Definition 16.16. *Let X be a variety, over an algebraically closed field of characteristic zero. We say that X is **rationally connected** if for two general points x and y of X, we may find a rational curve connecting x and y.*

One convenient way to restate this condition, is that for two general points x and y, we may find a morphism

$$f\colon \mathbb{P}^1_k \longrightarrow X,$$

such that $f(0)=x$ and $f(\infty)=y$. Indeed the nonconstant image of $\mathbb{P}^1_k$ is always birational to $\mathbb{P}^1_k$.

It is interesting to consider what happens in higher dimension. The space of cubic fourfolds $X \subset \mathbb{P}^5$ is a copy of projective space of dimension

$$\binom{3+5}{3} - 1 = 55.$$

Every cubic fourfold is unirational. Some cubic fourfolds are rational. For example, it is possible to write down smooth cubic fourfolds X which contain a pair of skew planes, L and M. This defines a rational map

$$\phi \colon L \times M \dashrightarrow X,$$

which assigns to a point (p, q) the point of intersection of the line $\langle p, q\rangle$ with $X - (L \cup M)$. The rational map

$$\psi \colon X \dashrightarrow L \times M,$$

which assigns to every point r the point (p, q), where p is the intersection of the 3-plane $\langle r, M\rangle$ with L and q is the intersection of the 3-plane $\langle r, L\rangle$ with M, is the inverse of ϕ. Thus X is birational to

$$L \times M = \mathbb{P}^2 \times \mathbb{P}^2 \simeq \mathbb{P}^4$$

The locus of cubics which contain a pair of skew planes has codimension two in $\mathbb{P}^{55}$. One can also write down other configurations of subvarieties of X which guarantee that X is rational.

Conjecture 16.17. *The locus of smooth rational cubic fourfolds inside the open subset of $\mathbb{P}^{55}$ consisting of all smooth cubics is a countable union of closed subsets of codimension two.*

In particular there are smooth irrational cubic fourfolds.

It is interesting to note the following

Theorem 16.18. *Fix a positive integer d.*

Then there is a positive integer n_0 such that if $n \geq n_0$ then every smooth hypersurface of degree d in $\mathbb{P}^n$ is unirational.

By contrast, there is no example of smooth hypersurface of degree $d \geq 4$ which is rational.