15. Cubics I

In this section we give a geometric application of some of the ideas of the previous sections. Recall the definition of a rational variety.

Definition 15.1. *A variety X over* $\operatorname{Spec} k$ *is* ***rational*** *if it birational to $\mathbb{P}^n_k$, for some n.*

Theorem 15.2. *Every smooth cubic $C \subset \mathbb{P}^2$ is irrational.*

We will prove (15.2) later.

Theorem 15.3. *Every smooth cubic surface $S \subset \mathbb{P}^3$ is rational.*

The key to the proof of (15.3) is the following celebrated:

Theorem 15.4. *Every smooth cubic surface $S \subset \mathbb{P}^3$ contains twenty seven lines.*

Example 15.5. *Let $S \subset \mathbb{P}^3$ be the cone over a cubic curve $C \subset \mathbb{P}^2$. Then S contains infinitely many lines.*

Lemma 15.6. *Every cubic surface $S \subset \mathbb{P}^3$ contains a line.*

Proof. A cubic is specified by choosing the coefficients of a homogeneous cubic in four variables of which there are $\binom{6}{3} = 20$; the space of all cubics is therefore naturally parametrised by $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{\, (l, F) \in \mathbb{G}(1,3) \times \mathbb{P}^{19} \,|\, l \subset V(F) \,\} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19}.$$

This is a closed subset of $\mathbb{G}(1,3) \times \mathbb{P}^{19}$ and the two natural projections $f \colon \Sigma \longrightarrow \mathbb{G}(1,3)$ and $g \colon \Sigma \longrightarrow \mathbb{P}^{19}$ are proper, since they are projective.

Let $l \in \mathbb{G}(1,3)$ and consider $f^{-1}(l)$. This is the space of cubics containing the line l. There are two ways to figure out what the fibre looks like.

One can change coordinates so that $l = V(X_2, X_3)$, so that the points of l are $[a : b : 0 : 0]$. In this case the coefficients of X^3, X^2Y, XY^2 and Y^3 must all vanish. The fibre is a copy of a linear subspace of dimension 15 in $\mathbb{P}^{19}$.

Aliter: Pick four distinct points p_1, p_2, p_3 and p_4 of l. Suppose $F(p_i) = 0$, for $1 \leq i \leq 4$. Then $F|_l$ is a cubic polynomial in two variables, vanishing at four points. Thus $F|_l$ is the zero polynomial. It follows that $l \subset V(F)$ if and only if F vanishes at p_i, for $1 \leq i \leq 4$.

The condition that $F(p_i) = 0$ imposes one linear constraint. One can check that these four points impose independent conditions, so that that the space of cubics containing all four points is a linear subspace of dimension 15.

Either way, Σ fibres over an irreducible base with irreducible fibres of the same dimension. It follows that Σ is irreducible of dimension $4 + 15 = 19$. It suffices then to exhibit a single cubic with finitely many lines, since then the morphism g must be dominant, whence surjective. It is a fun exercise to compute the twenty seven lines on $X^3 + Y^3 + Z^3 + T^3 = 0$. □

Lemma 15.7. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $l \subset S$ is a line then there are ten lines meeting l.*

In particular S contains two skew lines.

Proof. Consider the planes $H \subset \mathbb{P}^3$ containing l. Then $H \cap S = l \cup C$, where $C \subset H \simeq \mathbb{P}^2$ is a plane curve of degree two.

First observe that C is never a double line n. Indeed, if F and G are the linear polynomials which define l and F and H are the linear polynomials defining n, so that $F = 0$ is the plane spanned by l and n, then the equation of S has the form

$$FQ + GH^2,$$

for some quadratic polynomial Q. But then S is singular at the two points where $F = H = Q = 0$ (just compute partials).

Suppose that m is a line that intersects l. Then $C = m \cup n$, where n is another line, which also meets l. Thus lines that intersect l come in concurrent pairs and we just have to show that there are five such pairs.

We may suppose that l is given by $Z = T = 0$. Then S is defined by an equation of the form

$$AX^2 + 2BXY + CY^2 + 2DX + 2EY + F,$$

where A, B, C, D, E and F are homogeneous polynomials in Z and T.

The pencil of planes containing l is given by $Z = \lambda T$. Note that $C = C_\lambda$ is a pair of lines if and only if C is singular. C_λ is singular if and only if the determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

is zero. The determinant is a homogeneous polynomial of degree 5 in Z and T and so it suffices to show it has no repeated roots.

Supppose that $Z = 0$ is a root. There are two cases. If the singular point s of C_0 is not a point of l then we may assume that C_0 is given by $XY = 0$. Then every entry of the matrix above is divisible by Z, except B. On the other hand, as s is not a singular point of S it follows that Z^2 does not divide F. Thus Z^2 does not divide the determinant.

If s is a point of l then we may assume that C_0 is given by $X^2 - T^2 = 0$ and one can check that Z^2 does not divide the determinant. $\square$

Proof of (15.4). We just prove that S contains a pair of skew lines. (15.6) implies that S contains at least one line l. (15.7) implies that there are ten other lines meeting l. Pick one of them l'. Of the ten lines meeting l', at most one of them intersects l. Thus we may find a line m meeting l' not intersecting l. $\square$

Proof of (15.3). By assumption S contains two skew lines l and m. Define a rational map

$$\phi \colon l \times m \dashrightarrow S,$$

by sending the point (p, q) to the intersection of the line $n = \langle p, q \rangle$ with $S - (l \cup m)$. Since a cubic intersects a typical line in three points, and the line n intersects S at $p \in l$ and $q \in m$, there is an open subset of $l \times m$ such that the line n intersects S at one further point $r = \phi(p, q)$.

Define a rational map

$$\psi \colon S \dashrightarrow l \times m,$$

by sending $r \in S - (l \cup m)$ to (p, q), where p is the intersection point of the plane $\langle p, m \rangle$ with l and q is the intersection point of the plane $\langle p, l \rangle$ with m.

It is easy to check that ϕ and ψ are inverse. It follows that ϕ is birational. As $\mathbb{P}^1 \times \mathbb{P}^1 \simeq l \times m$ is rational, S is rational. $\square$