

14. TRANSCENDENCE DEGREE

Example 14.1. \mathbb{P}^n has dimension n . More generally a toric variety containing a torus \mathbb{G}_m^n has dimension n . In particular the toric variety corresponding to a fan F in N is equal to the rank of the free abelian group N .

Consider $\mathbb{G}(k, n)$. Then this contains an open subset U isomorphic to $\mathbb{A}^{(k+1)(n-k)}$. So $\mathbb{G}(k, n)$ has dimension $(k+1)(n-k)$. For example, $\mathbb{G}(1, 3)$ has dimension $2 \cdot 2 = 4$.

Suppose that X and Y are quasi-projective varieties. Then the dimension of $X \times Y$ is the sum of the dimensions.

We can use (??) to calculate the dimension using different methods. One way is to project onto a linear subspace. If we repeatedly project from a point (which is the same as projecting once from a linear space of positive dimension) then the induced morphism $X \rightarrow \mathbb{P}^k$ will eventually become dominant. At this point the morphism is finite over an open subset and the dimension of X is then k . Note that if we go back one step, then the closure of the image of X will be a hypersurface in \mathbb{P}^{k+1} .

Equivalently, if $X \subset \mathbb{P}^n$ and X has dimension d then a general linear space of dimension $n-d-1$ is disjoint from X and a general linear space of dimension $n-d$ meets X in a finite set of points. Note that general means that the linear space belongs to an open set of the corresponding Grassmannian. If X is closed, we can do slightly better, since if X is closed of dimension d , then every linear space of dimension $n-d$ must intersect X .

To calculate the dimension of an algebraic variety one can also use:

Definition 14.2. Let L/K be a field extension. The **transcendence degree** of L/K is equal to the supremum of the length x_1, x_2, \dots, x_k of algebraically independent elements of L/K .

It is easy to prove:

Theorem 14.3. Let X be an irreducible quasi-projective variety.

Then the dimension of X is equal to the transcendence degree of $K(X)/K$.

One trick to calculate dimensions is to use the generic point of a variety. If we have a morphism $\pi: X \rightarrow Y$ of irreducible varieties then μ_0 is actually the dimension of the generic fibre X_η , over the residue field of the generic point η of Y . Indeed the generic point ξ of X maps to the generic point of Y and so ξ is also the generic point of the generic fibre. The dimension of the generic fibre is the transcendence degree

of the residue field of ξ over the residue field of η . The dimension of X is the transcendence degree of the residue field of ξ over K . But the transcendence degree is additive on extensions.

Perhaps an easy example will make all of this clear. Consider \mathbb{A}_K^2 . Suppose the generic point is ξ , with residue field $K(x, y)$. This has transcendence degree two over K . If we take a projection down to \mathbb{A}_K^1 , with generic point η and residue field $K(y)$ then the transcendence degree of $K(x, y)/K(y)$ is one, the dimension of the generic fibre. $K(y)/K$ also has transcendence degree one and \mathbb{A}_K^1 has dimension one, as expected.

Now let's turn to calculating the dimension of some more examples, using these new techniques. Let us first calculate the dimension of the universal family over the Grassmannian.

$$\begin{array}{ccc} \Sigma & \xrightarrow{q} & \mathbb{P}^n \\ \downarrow p & & \\ \mathbb{G}(k, n) & & \end{array}$$

Note that there are two ways to proceed; we can either use the morphism p or q .

First we use the morphism p . If we fix an element $[\Lambda] \in \mathbb{G}(k, n)$ then the fibre of p will be a copy of the k -plane Λ . Thus every fibre of p is isomorphic to \mathbb{P}^k . It follows that Σ has dimension $k + (k + 1)(n - k)$.

Now let us use the morphism q . If we fix point $x \in \mathbb{P}^n$, then the fibre of q is equal to the set of k -planes in \mathbb{P}^n , containing x . This is isomorphic to a Grassmannian $\mathbb{G}(k - 1, n - 1)$. Thus the dimension of Σ is equal to $n + k(n - k)$, which is easily seen to be equal to the previous expression.

Note that also we can prove that Σ is irreducible. Either way, it fibres over an irreducible base, with irreducible fibres of the same dimension.

Similarly the universal family of conics has dimension six (=five+one=two+four) and this space is irreducible. It is perhaps more interesting to figure out the dimension of the secant variety and the space of incident l -planes to an irreducible projective variety $X \subset \mathbb{P}^n$.

First the space $\mathcal{C}_l(X)$ of l -planes which meets a closed subset X of \mathbb{P}^n . In this case the universal family over $\mathcal{C}_l(X)$ has dimension equal to

$$\dim X + l(n - l),$$

where the second factor is equal to the dimension of the space of l -planes which contains a point. Since we have already seen that this is a variety isomorphic to $\mathbb{G}(l - 1, n)$, it follows that the universal family is irreducible, provided X is irreducible.

In particular suppose that X has dimension k , and suppose that $l \leq n - k - 1$. Then a typical l -plane which meets X , will only meet X in one point. Thus the map from the universal family to $\mathbb{G}(l, n)$ is in fact birational, and the dimension of $\mathcal{C}_l(X)$ is

$$k + l(n - 1).$$

In other words the codimension of $\mathcal{C}_l(X)$ is

$$n - l - k.$$

Thus if $l = n - k - 1$, $\mathcal{C}_l(X)$ is a hypersurface in $\mathbb{G}(l, n)$.

Question 14.4. *Fix d . What is the smallest positive integer k such that any polynomial $f(x)$ of degree d over the field \mathbb{C} is a sum of k d th powers of linear forms?*

One way to answer this problem is to use the secant variety to the rational normal curve of degree d . Let V be a two dimensional complex vector space. Then $\mathbb{P}^1 = \mathbb{P}(V)$ and the rational normal curve is the set of pure d th powers in the vector space $\mathbb{P}^d = \mathbb{P}(\text{Sym}^d V)$. A polynomial $f(x)$ of degree d corresponds to a point of \mathbb{P}^d and it is a sum of k d th powers if and only if belongs to the locus of $k - 1$ -planes which intersect C in k points. We want to know when this locus is the whole of \mathbb{P}^n . In this case its dimension is n .

It turns out that even when look at the locus of secant lines that this problem is very hard for a general variety X . In general, we have a rational morphism

$$X \times X \dashrightarrow \mathbb{G}(1, n)$$

Now note that if $l \subset \mathbb{G}(1, n)$ is a point of the image, then this map is not finite over l if and only if l is contained in X . Since the only subvariety with the property that the line through every two points is contained in the subvariety, is a linear space, we may assume that this map has finite fibres over an open set of the image. Then the image has dimension $2k$, where k is the dimension of X . Then the universal family over the image, has dimension $2k + 1$ and the dimension of the image in \mathbb{P}^n then has dimension $2k + 1$ as well, provided that through a general point of the secant variety (the closure of the set of lines that meet X in at least two points), there passes only finitely secant lines.

Thus the expected dimension of the secant variety is $2k + 1$, provided this dimension is at most n . For example, the secant variety to a space curve is expected to be the whole of \mathbb{P}^3 and the secant variety to a surface in \mathbb{P}^5 is expected to be the whole of \mathbb{P}^5 .

Definition 14.5. *Let X be a closed irreducible non-degenerate (that is, X is not contained in a proper linear subspace) subvariety of \mathbb{P}^n .*

*The **deficiency of X** , denoted $\delta(X)$, is equal to the dimension of the family of secant lines passing through a general point of the secant variety.*

We have already seen then that the dimension of the secant variety is equal to $2k + 1 - \delta(X)$.

Let us calculate the secant variety to the d -uple embedding, at least in characteristic zero. Recall that if $X = \mathbb{P}(V) = \mathbb{P}^k$ then X is embedded in $\mathbb{P}(\text{Sym}^d(V))$, as the space of rank one symmetric tensors (the pure powers). The secant variety then consists of all rank at most two symmetric tensors, that is, anything which is a sum of two rank one symmetric tensors.

In the case of the Veronese, we get the space of rank two quadratic forms. As there are quadratic forms of rank three, it follows that the secant variety to the Veronese is a proper subset of \mathbb{P}^5 . In fact the space of rank two symmetric tensors is a hypersurface in \mathbb{P}^5 , given as the vanishing of a determinant. Expanding it follows that the secant variety is defined by a cubic polynomial. Note that the deficiency is equal to 1 in this case.

It is interesting to look at the dimension of some more exotic schemes. $\text{Spec } \mathbb{Z}$ has dimension one. Consider $\mathbb{A}_{\mathbb{Z}}^1$. This has dimension one over $\text{Spec } \mathbb{Z}$ and absolute dimension two. Consider $\mathbb{A}_{\mathbb{Z}}^2$. This has dimension two over $\text{Spec } \mathbb{Z}$ and so it has absolute dimension three.