13. Dimension of schemes

Our aim in this section is give a formal definition of the dimension of a variety, to compute the dimension in specific examples and to prove some of the interesting properties of the dimension.

Definition 13.1. *Let X be a topological space.*

The ***dimension of*** *X is equal to the supremum of the length n of strictly increasing sequences of irreducible closed subsets of X,*

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n.$$

We will call a chain ***maximal*** *if it cannot be extended to a longer chain.*

Note that if X is Noetherian then the dimension of X is, by definition, equal to the maximal dimension of an irreducible component. Note that also that the dimension of X is equal to the dimension of any dense open subset, and that the dimension of any subset is at most the dimension of X.

In general this notion of dimension is a little unwieldy, even for Noetherian topological spaces (in fact, it is pretty clear that this definition is useless for any topological space that is not Noetherian or at least close to Noetherian).

For quasi-projective varieties it is much better behaved. For example,

Theorem 13.2. *Let X be a quasi-projective variety.*

Then the dimension of X is equal to the length of any maximal chain of irreducible subvarieties.

Definition 13.3. *Let $f\colon X \longrightarrow I$ be a map from a topological space to an ordered set I. We say that f is* ***upper semi-continuous****, if for every $a \in I$, the set*

$$\{\, x \in X \mid f(x) \geq a \,\},$$

is closed in X.

The key result is:

Theorem 13.4. *Let $\pi\colon X \longrightarrow Y$ be a dominant morphism of quasi-projective varieties. Then the function*

$$\mu\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\mu(p)$ is the local dimension of the fibre $X_p = \pi^{-1}(\pi(p))$ at p. Moreover if X_0 is any irreducible component of X and Y_0 is the closure of the image, we have

$$\dim(X_0) = \dim(Y_0) + \mu_0,$$

where μ_0 is the minimum value of μ on X_0.

Note that semi-continuity of μ is equivalent to saying that the dimension can jump up on closed subsets, but not down. For example, consider what happens for the blow up of a point. In this case, μ is equal to zero outside of the exceptional divisor and it jumps up to one on the exceptional divisor.

We will prove these two results in tandem. Let $d = \dim X$. We will need an intermediary result, which is of independent interest:

Lemma 13.5. *Assume* $(13.2)_d$.

If $X \subset \mathbb{P}^n$ is a closed subset of dimension d and $H \subset \mathbb{P}^n$ is a hypersurface then

$$\dim(X \cap H) \geq \dim(X) - 1,$$

with equality if and only if $H \cap X$ does not contain a component of X of maximal dimension.

Proof. We might as well assume that X is irreducible and that H does not contain X. Pick a maximal chain of irreducible subvarieties of X which contains a component Y of $X \cap H$,

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_e.$$

Then $X = Z_e$ and $Y = Z_i$, some i. As we are assuming $(13.2)_d$, $d = e$ and $\dim Y = i$.

Suppose $Z \neq X$ is irreducible and

$$Y \subset Z \subset X.$$

I claim that $Z = Y$. To see this, if we pass to an open affine subset then Z and Y are defined by ideals $J \subset I \subset A$, where A is the coordinate ring, $I = \langle f \rangle$ is principal and J is a prime ideal. Pick $g \in J$, $g \neq 0$. Write $g = g_1 g_2 \dots g_k$ as a product of irreducibles. As J is a prime ideal, $g_i \in J$ for some i. As $g_i \in I$, $g_i = uf$, and u must be a unit as g_i is irreducible. But then $I = J$ and $Z = Y$.

It follows that $i = d - 1$ and so $\dim Y = d - 1$. □

Lemma 13.6. $(13.2)_{d-1}$ *implies* $(13.4)_d$.

Proof. The result is local on X, so we might as well assume that X and Y are irreducible and affine. We first show that

$$\mu(p) \geq \dim(X) - \dim(Y),$$

for every point of $p \in X$. If $e = \dim(Y) = \dim(X) = d$ there is nothing to prove. So we may assume that $e = \dim(Y) < d = \dim(X)$. Let $q = \pi(p)$. By (13.5) we may embed $Y \subset \mathbb{A}^n$ and pick a hyperplane $q \in H \subset Y$ such that $\dim(H \cap Y) = \dim(Y) - 1$. By an obvious induction, we may pick $\dim(Y)$ hyperplanes $H_1, H_2, \dots, H_e$, whose intersection is a finite set containing q. Working locally about q, we may assume that q is the only point in the intersection. Let $f_1, f_2, \dots, f_e$ be the corresponding polynomials. Then the fibre X_p is defined by the polynomials $g_1, g_2, \dots, g_e$, where $g_i = \pi^* f_i$. So

$$\dim(X_p) \geq \dim(X) - \dim(Y),$$

as required.

To finish the proof, by Noetherian induction applied to X, it suffices to prove that there is an open subset U of X such that

$$\mu(p) \leq \dim(X) - \dim(Y),$$

for every $p \in U$. As usual, we may assume that $X \subset Y \times \mathbb{A}^n$ and that π is projection onto the second factor. Factoring π into the product of n projections, we may assume that $n = 1$, by induction on n. We may assume that $X \subset Y \times \mathbb{A}^1$ is closed. If $X = Y \times \mathbb{A}^1$ then $\mu_0 = 1$ and it is clear that $\dim X \geq \dim Y + 1$. As we have already proved the reverse inequality, $\dim X = \dim Y + 1$.

Otherwise there is a fibre of dimension zero. As X is a proper subset of Y, $\dim X = \dim Y$ and $\mu_0 = 0$. Working locally, we may assume that X is defined by polynomials of the form $F \in A(Y)[S, T]$. Further there is a polynomial $F \in A(Y)[S, T]$ vanishing on X, such that F_y is not the zero polynomial, for at least one $y \in Y$. In this case, the set of points where F_y is not the zero polynomial, is an open subset of Y, and for any point in this open subset, the fibre has dimension zero. □

Lemma 13.7. $(13.4)_d$ *implies* $(13.2)_d$.

Proof. We may assume that X is affine. Pick a finite projection down to $\mathbb{A}^n$. As we are assuming $(13.4)_d$, $n = d$. It clearly suffices to prove the result for $X = \mathbb{A}^d$. Consider projection down to $\mathbb{A}^{d-1}$. Given a maximal chain of irreducible subsets

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n = \mathbb{A}^d,$$

let

$$\varnothing \neq Y_0 \subset Y_1 \subset \cdots \subset Y_n = \mathbb{A}^{d-1},$$

be the image in $\mathbb{A}^{d-1}$. Then there is an index i such that Z_i contains the general fibre and Z_{i-1} does not contain the general fibre. Other than that, Y_j determines Z_j and the result follows by induction on d. □

Proof of (13.2) *and* (13.4). Immediate from (13.6) and (13.7). □

Corollary 13.8. *Let $\pi\colon X \longrightarrow Y$ be a surjective and projective morphism of quasi-projective varieties. Then the function*

$$\lambda\colon Y \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(q)$ is the dimension of the fibre $X_q = \pi^{-1}(q)$ at q. Moreover if X_0 is any irreducible component of X, with image Y_0, then we have

$$\dim(X_0) = \dim(Y_0) + \lambda_0,$$

where λ_0 is the minimum value of λ on Y_0.

Proof. π is proper as it is projective. Therefore the set

$$\{\, y \in Y \mid \lambda(y) \geq k \,\},$$

is closed as it is the image of the set

$$\{\, x \in X \mid \mu(x) \geq k \,\},$$

which is closed by (13.4). □

Note that we cannot discard the hypothesis that π is projective in (13.8). For example, let X be the disjoint union of $\mathbb{A}^2$ minus the y-axis and a single point p. Define a morphism $\pi\colon X \longrightarrow Y = \mathbb{A}^1$ by sending the extra point to the origin and otherwise taking the projection onto the x-axis. Then the fibre dimension is one at every point of Y, other than at the origin, where it is zero. In particular λ is not upper semi-continuous in this example. On the other hand, μ is upper semi-continuous, by virtue of the fact that the extra point is isolated in X.

One rather beautiful consequence of (13.4) is the following:

Corollary 13.9. *Let $\pi\colon X \longrightarrow Y$ be a morphism of projective varieties.*

If Y is irreducible and every fibre of π is irreducible and of the same dimension, then X is irreducible.

Proof. Let $X = X_1 \cup X_2 \cup \cdots \cup X_k$ be the decomposition of X into its irreducible components. Let $\pi_i = \pi|_{X_i}\colon X_i \longrightarrow Y_i$, where Y_i is the image of X_i and let $\lambda_i\colon X_i \longrightarrow \mathbb{N}$ be the function associated to π_i, as in (13.8). Let

$$Z_i = \{\, y \in Y_i \mid \lambda_i(y) \geq \lambda_0 \,\}.$$

(13.8) implies that the closed sets $Z_1, Z_2, \dots, Z_k$ cover Y. As Y is irreducible it follows that there is an index i, say $i = 1$, such that $Z_1 = Y_1 = Y$. But then the fibres of π_1 and π are equal, as they are of the same dimension and the fibres of π are irreducible. This is only possible if $X = X_1$. □