12. Coherent Sheaves

Definition 12.1. *If $(X, \mathcal{O}_X)$ is a locally ringed space, then we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is* ***locally free*** *if there is an open affine cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is isomorphic to a direct sum of copies of $\mathcal{O}_{U_i}$. If the number of copies r is finite and constant, then $\mathcal{F}$ is called* ***locally free of rank*** *r.*

A ***sheaf of ideals*** *$\mathcal{I}$ is any subsheaf of $\mathcal{O}_X$.*

Definition 12.2. *Let $X = \operatorname{Spec} A$ be an affine scheme and let M be an A-module. $\tilde{M}$ is the $\mathcal{O}_X$-module which assigns to every open subset U_f the module M_f.*

Remark 12.3. *To realise (12.2) as a sheaf, note that $\tilde{M}(U)$ is the set of functions*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}},$$

which can be locally represented at $\mathfrak{p}$ as a/g, $a \in M$, $g \in R$, $\mathfrak{p} \notin U_g \subset U$.

Definition 12.4. *An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called* ***quasi-coherent*** *if there is an open cover $\{U_i = \operatorname{Spec} A_i\}$ by affines and isomorphisms $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$. If in addition M_i is a finitely generated A_i-module then we say that $\mathcal{F}$ is* ***coherent****.*

Proposition 12.5. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U = \tilde{M}$. If in addition X is noetherian then $\mathcal{F}$ is coherent if and only if M is a finitely generated A-module.*

Theorem 12.6. *Let $X = \operatorname{Spec} A$ be an affine scheme.*

The assignment $M \longrightarrow \tilde{M}$ defines an equivalence of categories between the category of A-modules to the category of quasi-coherent sheaves on X, which respects exact sequences, direct sum and tensor product, and which is functorial with respect to morphisms of affine schemes, $f\colon X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} B$. If in addition A is noetherian, this functor restricts to an equivalence of categories between the category of finitely generated A-modules to the category of coherent sheaves on X.

Theorem 12.7. *Let X be a scheme. Suppose that we are given a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

(1) $\mathcal{G}$ is quasi-coherent if and only if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent.
(2) If X is noetherian then $\mathcal{G}$ is coherent if and only if $\mathcal{F}$ and $\mathcal{H}$ are coherent.

Proof. Since this result is local, we may assume that $X = \operatorname{Spec} A$ is affine. The only non-trivial thing is to show that if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent then so is $\mathcal{G}$. By (II.5.6) of Hartshorne, there is an exact sequence on global sections,

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

It follows that there is a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{H} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0, \end{array}$$

whose rows are exact. By assumption, the first and third vertical arrow are isomorphisms, and the 5-lemma implies that the middle arrow is an isomorphism. □

Lemma 12.8. *Let $f\colon X \longrightarrow Y$ be a scheme.*

(1) *If $\mathcal{G}$ is a quasi-coherent sheaf (respectively X and Y are noetherian and $\mathcal{G}$ is coherent) on X then $f^*\mathcal{G}$ is quasi-coherent (respectively coherent).*
(2) *If $\mathcal{F}$ is a quasi-coherent sheaf on Y and either f is compact and separated or X is noetherian (respectively X and Y are noetherian, $\mathcal{F}$ is coherent and f is proper) then $f_*\mathcal{F}$ is quasi-coherent (respectively coherent).*

Definition-Lemma 12.9. *Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{I}$ is a quasi-coherent sheaf, which is coherent if X is noetherian. Moreover $\mathcal{I}$ defines a closed subscheme Y of X and there is a short exact sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Conversely, if $Y \subset X$ is a closed subscheme, then the kernel of the morphism of sheaves

$$\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

defines an ideal sheaf $\mathcal{I}_Y$, called the ***ideal sheaf of*** *Y* ***in*** *X.*

Remark 12.10. *Let $i\colon Y \longrightarrow X$ be a closed subscheme. If $\mathcal{F}$ is a sheaf on Y, then $\mathcal{G} = i_*\mathcal{F}$ is a sheaf on X, whose support is contained in Y. Conversely, given any sheaf $\mathcal{G}$ on X, whose support is contained in Y, then there is a unique sheaf $\mathcal{F}$ on Y such that $i_*\mathcal{F} = \mathcal{G}$.*

For this reason, it is customary, as in (12.9)*, to abuse notation, and to not distinguish between sheaves on Y and sheaves on X, whose support is contained in Y.*

Most of what we have done with algebras and modules, makes sense for graded algebras and graded modules, in which case we get sheaves on Proj of the graded ring.

Definition 12.11. *Let S be a graded ring and let M be a graded S-module. If $\mathfrak{p} \triangleleft S$ is a homogeneous ideal, then $M_{(\mathfrak{p})}$ denotes those elements of the localisation $M_{\mathfrak{p}}$ of degree zero.*

$\tilde{M}$ is the sheaf on $\operatorname{Proj} S$, which given an open subset $U \subset \operatorname{Proj} S$, then $\tilde{M}(U)$ denotes those functions

$$s \colon U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})},$$

which are locally fractions of degree zero.

Proposition 12.12. *Let S be a graded ring, let M be a graded S-module and let $X = \operatorname{Proj} S$.*

(1) For any $\mathfrak{p} \in X$, $(\tilde{M})_{\mathfrak{p}} \simeq M_{(\mathfrak{p})}$.
(2) If $f \in S$ is homogeneous,

$$(\tilde{M})_{U_f} \simeq \tilde{M}_{(f)}.$$

(3) $\tilde{M}$ is a quasi-coherent sheaf. If S is noetherian and M is finitely generated then $\tilde{M}$ is a coherent sheaf.

Definition 12.13. *Let $X = \operatorname{Proj} S$, where S is a graded ring. If n is any integer, then set*

$$\mathcal{O}_X(n) = S(n)\tilde{}.$$

If $\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules,

$$\mathcal{F}(n) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n).$$

Let

$$\Gamma_*(X, \mathcal{F}) = \bigoplus_{m \in \mathbb{N}} \Gamma(X, \mathcal{F}(n)).$$

Lemma 12.14. *Let S be a graded ring, $X = \operatorname{Proj} S$ and let M be a graded S-module.*

(1) $\mathcal{O}_X(n)$ is an invertible sheaf.
(2) $\tilde{M}(n) \simeq M(n)\tilde{}$. In particular $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$.
(3) Formation of the twisting sheaf is functorial with respect to morphisms of graded rings.

Proposition 12.15. *Let A be a ring, let $S = A[x_0, x_1, \dots, x_r]$ and let $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, x_1, \dots, x_r]$.*

Then

$$\Gamma_*(X, \mathcal{O}_X) \simeq S.$$

Lemma 12.16. *Let S be a graded ring, generated as an S_0-algebra by S_1.*

If $X = \operatorname{Proj} S$ and $\mathcal{F}$ is a quasi-coherent sheaf on X, then

$$\Gamma_*(X, \mathcal{F})\tilde{} = \mathcal{F}.$$

Theorem 12.17. *Let A be a ring.*

(1) *If $Y \subset \mathbb{P}^n_A$ is a closed subscheme then $Y = \operatorname{Proj} S/I$, for some homogeneous ideal $I \lhd S = A[x_1, x_2, \dots, x_n]$.*
(2) *Y is projective over $\operatorname{Spec} A$ if and only if it is isomorphic to $\operatorname{Proj} T$ for some graded ring, for which there are finitely many elements of T_1 which generate T as a $T_0 = A$-algebra.*

Proof. Let $\mathcal{I}_Y$ the ideal sheaf of Y in X. Then there is an exact sequence,

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Twisting by $\mathcal{O}_X(n)$ is exact (in fact $\mathcal{O}_X(n)$ is an invertible sheaf), so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Taking global sections is left exact, so we get an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{I}_Y(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)).$$

Taking the direct sum, there is therefore an injective map

$$I = \Gamma_*(X, \mathcal{I}_Y) = \Gamma_*(X, \mathcal{O}_X) \simeq S.$$

It follows that $I \lhd S$ is a homogeneous ideal. Let $\tilde{I}$ be the associated sheaf. Since $\mathcal{I}_Y$ is quasi-coherent, in fact $\tilde{I} = \mathcal{I}_Y$ (see Hartshorne (II.5.15)). But then the subscheme determined by I is equal to Y. Hence (1).

If Y is projective over $\operatorname{Spec} A$ then we may assume that $Y \subset \mathbb{P}^n_A$. By (1) $Y \simeq \operatorname{Proj} S/I$, and if $T = S/I$, then $T_0 \simeq A$ and the image of $x_0, x_1, \dots, x_n \in T_1$ generate T. Conversely, any such algebra is the quotient of S. The kernel I is a homogeneous ideal and $Y \simeq \operatorname{Proj} S/I$. □

Definition 12.18. *Let Y be a scheme. $\mathcal{O}_Y(1) = g^*\mathcal{O}_{\mathbb{P}^r}(1)$ is the sheaf on $\mathbb{P}^r_Y$, where $g \colon \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\operatorname{Spec} \mathbb{Z}}$ is the natural morphism.*

We say that a morphism $i \colon X \longrightarrow Z$ is an ***immersion*** *if i induces an isomorphism of X with a locally closed subset of Y.*

We say that an invertible sheaf $\mathcal{L}$ on a scheme X over Y is ***very ample*** *if there is an immersion $i \colon X \longrightarrow \mathbb{P}^r_Y$ over Y, such that $\mathcal{L} \simeq i^*\mathcal{O}_Y(1)$.*

Lemma 12.19. *Let X be a scheme over Y.*

Then X is projective over Y if and only if X is proper over Y and there is a very ample sheaf on X.

Proof. One direction is clear; if X is projective over Y, then it is proper and we just pullback $\mathcal{O}_Y(1)$.

If X is proper over Y then the image of X in $\mathbb{P}^r_Y$ is closed, and so X is projective over Y. □

Definition 12.20. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is* ***globally generated*** *if there are elements $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that for every point $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$, generate the stalk as an $\mathcal{O}_{X,x}$-module.*

Lemma 12.21. *Let X be a scheme and*

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(1) $\mathcal{F}$ is globally generated.
(2) The natural map

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \longrightarrow \mathcal{F},$$

is surjective.
(3) $\mathcal{F}$ is a quotient of a free sheaf.

Proof. Clear. □

Lemma 12.22 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module.*

$$f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} f^*\mathcal{G}) = f_*\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \mathcal{G}.$$

Theorem 12.23 (Serre)**.** *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

Then there is a positive integer $n_0 \geq 0$ such that $\mathcal{F}(n)$ is globally generated for all $n \geq n_0$.

Proof. By assumption there is a closed immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Let $\mathcal{G} = i_*\mathcal{F}$. Then (12.22) implies that

$$\mathcal{G}(n) = i_*\mathcal{F}(n).$$

Then $\mathcal{F}(n)$ is globally generated if and only if $\mathcal{G}(n)$ is globally generated. As i is a closed immersion it is a proper morphism; as $\mathcal{F}$ is coherent and i is proper, $\mathcal{G}$ is coherent. Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

Consider the standard open affine cover U_i, $0 \leq i \leq r$ of $\mathbb{P}^r_A$. Since $\mathcal{F}$ is coherent, $\mathcal{F}_i = \mathcal{F}|_{U_i} = \tilde{F}_i$, for some finitely generated

$A[x_0/x_i, x_1/x_i, \dots, x_r/x_i]$-module F_i. Pick generators s_{ij} of F_i. For each j, we may lift $x_i^{n_{ij}} s_{ij}$ to t_{ij}, for some n_{ij} (see (II.5.14)). By finiteness, we may assume that $n = n_{ij}$ does not depend on i and j. Now the natural map

$$x_i^n \colon \mathcal{F} \longrightarrow \mathcal{F}(n),$$

is an isomorphism over U_i. Thus t_{ij} generate the stalks of $\mathcal{F}$. □

Corollary 12.24. *Let X be a scheme projective over a noetherian ring A and let $\mathcal{F}$ be a coherent sheaf.*

Then $\mathcal{F}$ is a quotient of a direct sum of line bundles of the form $\mathcal{O}_X(n_i)$.

Proof. Pick $n > 0$ such that $\mathcal{F}(n)$ is globally generated. Then

$$\bigoplus_{i=1}^{k} \mathcal{O}_X \longrightarrow \mathcal{F}(n),$$

is surjective. Now just tensor by $\mathcal{O}_X(-n)$. □