11. Toric varieties

Definition 11.1. *A* ***fan*** *in $N_{\mathbb{R}}$ is a set F of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in F is a cone in F, and*
- *the intersection of any two cones in F is a face of each cone.*

It turns out that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{GL}(n, \mathbb{Z})$.

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan (see (2.12) of Hartshorne). It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 11.2. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$

One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\sigma \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality and the fact that $u \in S_\sigma$, we have $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. □

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 11.3. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_{\mathbb{R}} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ then coordinates on U_0 are $x = X/Y$ and coordinates on U_1 are $y = Y/X = 1/x$).*

Example 11.4. *Now suppose that we start with three cones in $N_{\mathbb{R}} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of $\mathbb{A}^2$.*

The dual cone of σ_1 is the cone spanned by f_1 and f_2 in $M_{\mathbb{R}} = \mathbb{R}^2$. The dual cone of σ_2 is the cone spanned by $-f_1$ and $-f_1 + f_2$. So we have $U_1 = \operatorname{Spec} K[x, y]$ and $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. On the other hand, if we start with $\mathbb{P}^2$ with homogeneous coordinates $[X : Y : Z]$ and the two basic open subsets $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ and $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, then we get the same picture, if we set $x = Y/X$, $y = Z/X$ (since then $X/Y = x^{-1}$ and $Z/Y = Z/X \cdot X/Y = yx^{-1}$). With a little more work one can check that we have $\mathbb{P}^2$.

Example 11.5. *More generally suppose we start with $n + 1$ vectors $v_1, v_2, \dots, v_{n+1}$ in $N_{\mathbb{R}} = \mathbb{R}^n$ which sum to zero such that the first n vectors $v_1, v_2, \dots, v_n$ span the standard lattice. Let F be the fan obtained by taking all the cones spanned by all subsets of at most n vectors. One can check that the resulting toric variety is $\mathbb{P}^n$.*

Example 11.6. *Now suppose that we take the four vectors e_1, e_2, $-e_1$ and $-e_2$ in $N_{\mathbb{R}} = \mathbb{R}^2$ and let F be the fan consisting of all cones spanned by at most two vectors (but not pairs of inverse vectors, that is, neither e_1 and $-e_1$ nor e_2 and $-e_2$). Then we get four copies of $\mathbb{A}^2$. It is easy to check that the resulting toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed the top two fans glue together to get $\mathbb{P}^1 \times \mathbb{A}^1$ and so on.*

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 11.7. *Let $S \subset M$ be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ is the multiplicative subsemigroup of K (and not the additive).

Proof. Suppose we are given a ring homomorphism

$$f\colon K[S] \longrightarrow K.$$

Define

$$g\colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. □

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0, 1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (11.7) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is, a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. Let $O_\sigma \subset U_\sigma$ be the orbit of x_σ and let $V(\sigma)$ be the closure of O_σ.

Example 11.8. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 11.9. *Let G and H be algebraic groups which act on varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{GL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 11.10. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

Note that this is an easy way to see the birational map between the quadric $Q \subset \mathbb{P}^3$ and $\mathbb{P}^2$ given by projection from a point.