HWK #5, DUE WEDNESDAY 02/12

1. Let l_1 and $l_2 \subset \mathbb{P}^3$ be two skew lines. Let

$$Q = \{\, [l] \in \mathbb{G}(1,3) \,|\, l \cap l_1, l \cap l_2 \neq \varnothing \,\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5.$$

Show that Q is a quadric surface, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ contained in some linear subspace $\mathbb{P}^3 \subset \mathbb{P}^5$. What happens if l_1 and l_2 are not skew?

2. Now let $Q \subset \mathbb{P}^3$ be a quadric surface of rank four. Show that the two families of lines on Q correspond to two families of conics on $\mathbb{G}(1,3)$ lying on two complementary planes Λ_1 and $\Lambda_2 \subset \mathbb{P}^5$. Show that conversely the lines in $\mathbb{P}^3$ corresponding to a conic lying in $\mathbb{G}(1,3)$ sweep out a quadric surface provided that the plane spanned by the conic does not lie in $\mathbb{G}(1,3)$. What happens to this correspondence if either the quadric has rank three or the plane lies in $\mathbb{G}(1,3)$?

3. (a) Let $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ be the conic given by $Z_1^2 - Z_0 Z_2 = Z_3 = 0$. Find equations for the locus of lines

$$\mathcal{C}_1(C) = \{\, [l] \in \mathbb{G}(1,3) \,|\, l \cap C \neq \varnothing \,\} \subset \mathbb{G}(1,3)$$

which meet C.

(b) Same question for the twisted cubic given as the image of $[S : T] \longrightarrow [S^3 : S^2T : ST^2 : T^3]$.