HWK #3, DUE WEDNESDAY 01/29

1. Let
$$\nu_{\alpha,\beta} \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
be the morphism
$$[S:T] \longrightarrow [S^4 - \beta S^3T : S^3T - \beta S^2T^2 : \alpha S^2T^2 - ST^3 : \alpha ST^3 - T^4],$$
where α and $\beta \in K$.
(i) Show that the curves $C_{\alpha,\beta} = \nu_{\alpha,\beta}(\mathbb{P}^1)$ are closed subsets.
(ii) Show that $C_{\alpha,\beta}$ lies on the surface $V(XW - YZ)$.
(iii) Show that $C_{\alpha,\beta}$ is the zero locus of a bihomogeneous polynomial of type $(1,3)$.
(iv) Show that $C_{\alpha,\beta}$ is the zero locus of one quadratic and two cubic polynomials.
2. Let p_1, p_2, p_3 and p_4 be points in $\mathbb{P}^1$, with coordinates z_1, z_2, z_3 and z_4. If the cardinality of the set
$$P = \{ p_1, p_2, p_3, p_4 \},$$
is at least three then the cross-ratio of p_1, p_2, p_3 and p_4 is
$$\lambda = \lambda(p_1, p_2, p_3, p_4) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \in \mathbb{P}^1.$$
(i) Show that the cross-ratio λ is invariant under the action of PGL(2) on the four ordered points.
(ii) Show that the only invariant of the four ordered points is given by the cross-ratio λ (in other words, show that any two ordered set of four points are projectively equivalent if and only if they have the same cross-ratio).
3. Show that there infinitely many of the curves $C_{\alpha,\beta}$ are not projectively equivalent.
4. Find the projection of the twisted cubic from $[1:0:0:1]$ and $[0:1:0:0]$. Show that, up to projective equivalence, these are the only two cases.
5. Show that $C_{\alpha,\beta}$ is the projection of a rational quartic curve in $\mathbb{P}^4$. Thereby show that projecting the same variety from different points we can get infinitely many projectively inequivalent varieties.
Challenge Problem:
6. Show that every square matrix over a field satisfies its characteristic polynomial. (*Hint: reduce to the case of an algebraically closed field.*

Show that the space of all matrices is naturally a variety and identify a large subset where the result is obvious).