## 7. Ample invertible sheaves

**Theorem 7.1.** *Let $X$ be a scheme over a ring $A$.*

(1) *If $\phi\colon X \longrightarrow \mathbb{P}^n_A$ is an $A$-morphism then $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf on $X$, which is generated by the global sections $s_0, s_1, \dots, s_n$, where $s_i = \phi^* x_i$.*

(2) *If $\mathcal{L}$ is an invertible sheaf on $X$, which is generated by the global sections $s_0, s_1, \dots, s_n$, then there is a unique $A$-morphism $\phi\colon X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i = \phi^* x_i$.*

*Proof.* It is clear that $\mathcal{L}$ is an invertible sheaf. Since $x_0, x_1, \dots, x_n$ generate the ring $A[x_0, x_1, \dots, x_n]$, it follows that $x_0, x_1, \dots, x_n$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$. Thus $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. Hence (1).

Now suppose that $\mathcal{L}$ is an invertible sheaf generated by $s_0, s_1, \dots, s_n$. Let

$$X_i = \{\, p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{L}_p \,\}.$$

Then $X_i$ is an open subset of $X$ and the sets $X_0, X_1, \dots, X_n$ cover $X$. Define a morphism

$$\phi_i\colon X_i \longrightarrow U_i,$$

where $U_i$ is the standard open subset of $\mathbb{P}^n_A$, as follows: Since

$$U_i = \operatorname{Spec} A[y_0, y_1, \dots, y_n],$$

where $y_j = x_j/x_i$, is affine, it suffices to give a ring homomorphism

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}).$$

We send $y_j$ to $s_j/s_i$, and extend by linearity. The key observation is that the ratio is a well-defined element of $\mathcal{O}_{X_i}$, which does not depend on the choice of isomorphisms $\mathcal{L}|_V \simeq \mathcal{O}_V$, for open subsets $V \subset X_i$.

It is then straightforward to check that the set of morphisms $\{\phi_i\}$ glues to a morphism $\phi$ with the given properties. $\square$

**Example 7.2.** *Let $X = \mathbb{P}^1_k$, $A = k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(2)$.*

*In this case, global sections of $\mathcal{L}$ are generated by $S^2$, $ST$ and $T^2$. This morphism is represented globally by*

$$[S : T] \longrightarrow [S^2 : ST : T^2].$$

*The image is the conic $XZ = Y^2$ inside $\mathbb{P}^2_k$.*

*More generally one can map $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$. More generally still, one can map $\mathbb{P}^m_k$ into $\mathbb{P}^n_k$ using the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(1)$.*

**Corollary 7.3.**

$$\operatorname{Aut}(\mathbb{P}^n_k) \simeq \operatorname{PGL}(n+1, k).$$

*Proof.* First note that $\mathrm{PGL}(n+1,k)$ acts naturally on $\mathbb{P}^n_k$ and that this action is faithful.

Now suppose that $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Let $\mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}^n_k}(1)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, it follows that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1)$. As $\mathcal{L}$ is globally generated, we must have $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $s_i = \phi^* x_i$. Then $s_0, s_1, \dots, s_n$ is a basis for the $k$-vector space $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. But then there is a matrix

$$A = (a_{ij}) \in \mathrm{GL}(n+1,k) \qquad \text{such that} \qquad s_i = \sum_{ij} a_{ij} x_j.$$

Since the morphism $\phi$ is determined by $s_0, s_1, \dots, s_n$, it follows that $\phi$ is determined by the class of $A$ in $\mathrm{GL}(n+1,k)$. □

**Lemma 7.4.** *Let $\phi\colon X \longrightarrow \mathbb{P}^n_A$ be an $A$-morphism. Then $\phi$ is a closed immersion if and only if*

*(1) $X_i = X_{s_i}$ is affine, and*
*(2) the natural map of rings*

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \qquad \textit{which sends} \qquad y_j \longrightarrow \frac{\sigma_j}{\sigma_i},$$

*is surjective.*

*Proof.* Suppose that $\phi$ is a closed immersion. Then $X_i$ is isomorphic to $\phi(X) \cap U_i$, a closed subscheme of affine space. Thus $X_i$ is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then $X_i$ is a closed subscheme of $U_i$ and so $X$ is a closed subscheme of $\mathbb{P}^n_A$. □

**Theorem 7.5.** *Let $X$ be a projective scheme over an algebraically closed field $k$ and let $\phi\colon X \longrightarrow \mathbb{P}^n_k$ be a morphism over $k$, which is given by an invertible sheaf $\mathcal{L}$ and global sections $s_0, s_1, \dots, s_n$ which generate $\mathcal{L}$. Let $V \subset \Gamma(X, \mathcal{L})$ be the space spanned by the sections.*

*Then $\phi$ is a closed immersion if and only if*

*(1) $V$ **separates points**: that is, given $p$ and $q \in X$ there is $\sigma \in V$ such that $\sigma \in \mathfrak{m}_P \mathcal{L}_p$ but $\sigma \notin \mathfrak{m}_q \mathcal{L}_q$.*
*(2) $V$ **separates tangent vectors**: that is, given $p \in X$ the set*

$$\{\, \sigma \in V \mid \sigma \in \mathfrak{m}_p \mathcal{L}_p \,\},$$

*spans $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$.*

*Proof.* Suppose that $\phi$ is a closed immersion. Then we might as well consider $X \subset \mathbb{P}^n_k$ as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of $\mathbb{P}^n_k$ which vanishes at $p$ but not at $q$ (equivalently pick a hyperplane which contains $p$ but not $q$).

Similarly linear functions on $\mathbb{P}^n_k$ separate tangent vectors on the whole of projective space, so they certainly separate on $X$.

Now suppose that (1) and (2) hold. Then $\phi$ is clearly injective. Since $X$ is proper over $\operatorname{Spec} k$ and $\mathbb{P}^n_k$ is separated over $\operatorname{Spec} k$ it follows that $\phi$ is proper. In particular $\phi(X)$ and $\phi$ is a homeomorphism onto $\phi(X)$. It remains to show that the map on stalks

$$\mathcal{O}_{\mathbb{P}^n_k,p} \longrightarrow \mathcal{O}_{X,x},$$

is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. □

**Definition 7.6.** *Let $X$ be a noetherian scheme. We say that an invertible sheaf $\mathcal{L}$ is* ***ample*** *if for every coherent sheaf $\mathcal{F}$ there is an integer $n_0 > 0$ such that $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^n$ is globally generated, for all $n \geq n_0$.*

**Lemma 7.7.** *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme. TFAE*

*(1) $\mathcal{L}$ is ample.*
*(2) $\mathcal{L}^m$ is ample for all $m > 0$.*
*(3) $\mathcal{L}^m$ is ample for some $m > 0$.*

*Proof.* (1) implies (2) implies (3) is clear.

So assume that $\mathcal{M} = \mathcal{L}^m$ is ample and let $\mathcal{F}$ be a coherent sheaf. For each $0 \leq i \leq m-1$, let $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{L}^i$. By assumption there is an integer $n_i$ such that $\mathcal{F}_i \otimes \mathcal{M}^n$ is globally generated for all $n \geq n_i$. Let $n_0$ be the maximum of the $n_i$. If $n \geq n_0 m$, then we may write $n = qm + i$, where $0 \leq i \leq m-1$ and $q \geq n_0 \geq n_i$.

But then

$$\mathcal{F} \otimes \mathcal{L}^m = \mathcal{F}_i \otimes \mathcal{M}^q,$$

which is globally generated. □

**Theorem 7.8.** *Let $X$ be a scheme of finite type over a Noetherian ring $A$ and let $\mathcal{L}$ be an invertible sheaf on $X$.*

*Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample for some $m > 0$.*

*Proof.* Suppose that $\mathcal{L}^m$ is very ample. Then there is an immersion $X \subset \mathbb{P}^r_A$, for some positive integer $r$, and $\mathcal{L}^m = \mathcal{O}_X(1)$. Let $\bar{X}$ be the closure. If $\mathcal{F}$ is any coherent sheaf on $X$ then there is a coherent sheaf $\overline{\mathcal{F}}$ on $\bar{X}$, such that $\mathcal{F} = \overline{\mathcal{F}}|_X$. By Serre's result, $\overline{\mathcal{F}}(k)$ is globally generated for all $k \geq k_0$, for some integer $k_0$. It follows that $\mathcal{F}(k)$ is globally generated, for all $k \geq k_0$, so that $\mathcal{L}^m$ is ample, and the result follows by (7.7).

Conversely, suppose that $\mathcal{L}$ is ample. Given $p \in X$, pick an open affine neighbourhood $U$ of $p$ so that $\mathcal{L}|_U$ is free. Let $Y = X - U$, give it the reduced induced strucure, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent. Pick $n > 0$ so that $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated. Then we may find $s \in \mathcal{I} \otimes \mathcal{L}^n$ not vanishing at $p$. We may identify $s$ with $s' \in \mathcal{O}_U$ and then $p \in U_s \subset U$, an affine subset of $X$.

By compactness, we may cover $X$ by such open affines and we may assume that $n$ is fixed. Replacing $\mathcal{L}$ by $\mathcal{L}^n$ we may assume that $n = 1$. Then there are global sections $s_1, s_2, \dots, s_k \in H^0(X, \mathcal{L})$ such that $U_i = U_{s_i}$ is an open affine cover.

Since $X$ is of finite type, each $B_i = H^0(U_i, \mathcal{O}_{U_i})$ is a finitely generated $A$-algebra. Pick generators $b_{ij}$. Then $s^n b_{ij}$ lifts to $s_{ij} \in H^0(X, \mathcal{L}^n)$. Again we might as well assume $n = 1$.

Now let $\mathbb{P}^N_A$ be the projective space with coordinates $x_1, x_2, \dots, x_k$ and $x_{ij}$. Locally we can define a map on each $U_i$ to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion. $\square$