5. The inverse function theorem

We now want to aim for a version of the Inverse function Theorem. In differential geometry, the inverse function theorem states that if a function is an isomorphism on tangent spaces, then it is locally an isomorphism. Unfortunately this is too much to expect in algebraic geometry, since the Zariski topology is too weak for this to be true. For example consider a curve which double covers another curve. At any point where there are two points in the fibre, the map on tangent spaces is an isomorphism. But there is no Zariski neighbourhood of any point where the map is an isomorphism.

Thus a minimal requirement is that the morphism is a bijection. Note that this is not enough in general for a morphism between algebraic varieties to be an isomorphism. For example in characteristic p, Frobenius is nowhere smooth and even in characteristic zero, the parametrisation of the cuspidal cubic is a bijection but not an isomorphism.

Lemma 5.1. *If $f\colon X \longrightarrow Y$ is a projective morphism with finite fibres, then f is finite.*

Proof. Since the result is local on the base, we may assume that Y is affine. By assumption $X \subset Y \times \mathbb{P}^n$ and we are projecting onto the first factor. Possibly passing to a smaller open subset of Y, we may assume that there is a point $p \in \mathbb{P}^n$ such that X does not intersect $Y \times \{p\}$.

As the blow up of $\mathbb{P}^n$ at p, fibres over $\mathbb{P}^{n-1}$ with fibres isomorphic to $\mathbb{P}^1$, and the composition of finite morphisms is finite, we may assume that $n = 1$, by induction on n.

We may assume that p is the point at infinity, so that $X \subset Y \times \mathbb{A}^1$, and X is affine. Now X is defined by $f(x) \in A(Y)[x]$, where the coefficients of $f(x)$ lie in $A(Y)$. Suppose that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

We may always assume that a_n does not vanish at y. Passing to the locus where a_n does not vanish, we may assume that a_n is a unit, so that dividing by a_n, we may assume that $a_n = 1$. In this case the ring B is a quotient of the ring

$$A[x]/\langle f \rangle.$$

But the latter is generated over A by 1, x, $\dots x^{n-1}$, and so is a finitely generated module over A. □

Theorem 5.2. *Let $f\colon X \longrightarrow Y$ be a projective morphism between quasi-projective varieties.*

Then f is an isomorphism if and only if it is a bijection and the differential df_p is injective.

Proof. One direction is clear. Otherwise assume that f is projective and a bijection on closed points. Then f is finite by (5.1). The result is local on the base, so we may assume that $Y = \operatorname{Spec} C$ is affine, in which case $X = \operatorname{Spec} D$ is affine, where C is a finitely generated D-module. Pick $x \in X$ and let $y = f(x)$. Then $x = \mathfrak{p}$ and $y = \mathfrak{q}$ are two prime ideals in C and D. Let A be the local ring of Y at y, B of X at x. Then A is the localisation of C at the multiplicative subset $S = C - \mathfrak{q}$ and as x is the unique point of the fibre, B is the localisation of D by the multiplicative subset $T = S \cdot D$, so that B is a finitely generated A-module.

Let $\phi \colon A \longrightarrow B$ be the induced ring homomorphism. Then B is a finitely generated A-module and we just need to show that ϕ is an isomorphism.

As f is a bijection on closed points, it follows that ϕ is injective. So we might as well suppose that ϕ is an inclusion. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}$ be the maximal ideal of B. By assumption

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2},$$

is surjective. But then

$$\mathfrak{m}B + \mathfrak{n}^2 = \mathfrak{n}.$$

By Nakayama's Lemma applied to the B-module $\mathfrak{n}/\mathfrak{m}B$, it follows that $\mathfrak{m}B = \mathfrak{n}$. But then

$$B/A \otimes A/\mathfrak{m} = B/(\mathfrak{m}B + A) = B/(\mathfrak{n} + A) = 0.$$

Nakayama's Lemma applied to the finitely generated A-module B/A implies that $B/A = 0$ so that ϕ is an isomorphism. □

Lemma 5.3. *Suppose that $X \subset \mathbb{P}^n$ is a quasi-projective variety and suppose that $\pi \colon X \longrightarrow Y$ is the morphism induced by projection from a linear subspace Λ.*

Let $y \in Y$. Then $\pi^{-1}(y) = \langle \Lambda, y \rangle \cap X$. If further this fibre consists of one point, then the map between Zariski tangent spaces is an isomorphism if the intersection of $\langle \Lambda, x \rangle$ with the Zariksi tangent space to X at x has dimension zero.

Proof. Easy. □

Proposition 5.4. *Let X be a smooth irreducible subset of $\mathbb{P}^n$ of dimension k. Consider the projection Y of X down to a smaller dimensional projective space $\mathbb{P}^m$, from a linear space Λ of dimension $n - m - 1$.*

If the dimension of $m \geq 2k+1$ *and* Λ *is general (that is, belongs to an appropriate open subset of the Grassmannian) then* π *is an isomorphism.*

Proof. Since projection from a general linear space is the same as a sequence of projections from general points, we may assume that Λ is in fact a point p, so that $m = n - 1$.

Now we know that π is a bijection provided that p does not lie on any secant line. Since the secant variety has dimension at most $2k + 1$, it follows that we may certainly find a point away from the secant variety, provided that $n > 2k + 1$. Now since a tangent line is a limit of secant lines, it follows that such a point will also not lie on any tangent lines.

But then π is then an isomorphism on tangent spaces, whence an isomorphism. □

For example, it follows that any curve may be embedded in $\mathbb{P}^3$ and any surface in $\mathbb{P}^5$.