3. Smoothness and the Zariski tangent space

We want to give an algebraic notion of the tangent space. In differential geometry, tangent vectors are equivalence classes of maps of intervals in $\mathbb{R}$ into the manifold. This definition lifts to algebraic geometry over $\mathbb{C}$ but not over any other field (for example a field of characteristic p).

Classically tangent vectors are determined by taking derivatives, and the tangent space to a variety X at x is then the space of tangent directions, in the whole space, which are tangent to X. Even is this is how we will compute the tangent space in general, it is still desirable to have an intrinsic definition, that is, a definition which does not use the fact that X is embedded in $\mathbb{P}^n$.

Now note first that the notion of smoothness is surely local and that if we want an intrinsic definition, then we want a definition that only uses the functions on X. Putting this together, smoothness should be a property of the local ring of X at p. On the other hand taking derivatives is the same as linear approximation, which means dropping quadratic and higher terms.

Definition 3.1. *Let X be a variety and let $p \in X$ be a point of X. The* ***Zariski tangent space*** *of X at p, denoted T_pX, is equal to the dual of the quotient*

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.

Note that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space. Suppose that we are given a morphism

$$f\colon X \longrightarrow Y,$$

which sends p to q. In this case there is a ring homomorphism

$$f^*\colon \mathcal{O}_{Y,q} \longrightarrow \mathcal{O}_{X,p}$$

which sends the maximal ideal $\mathfrak{n}$ into the maximal ideal $\mathfrak{m}$. Thus we get an induced map

$$df\colon \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2.$$

On the other hand, geometrically the map on tangent spaces obviously goes the other way. It follows that we really do want the dual of $\mathfrak{m}/\mathfrak{m}^2$. In fact $\mathfrak{m}/\mathfrak{m}^2$ is the dual of the Zariski tangent space, and is referred to as the *cotangent space*.

In particular, given a morphism $f\colon X \longrightarrow Y$ carrying p to q, then there is a linear map

$$df\colon T_pX \longrightarrow T_qY.$$

Definition 3.2. *Let X be a quasi-projective variety.*

*We say that X is **smooth** at p if the local dimension of X at p is equal to the dimension of the Zariski tangent space at p.*

Now the tangent space to $\mathbb{A}^n$ is canonically a copy of $\mathbb{A}^n$ itself, considered as a vector space based at the point in question. If $X \subset \mathbb{A}^n$, then the tangent space to X is included inside the tangent space to $\mathbb{A}^n$. The question is then how to describe this subspace.

Lemma 3.3. *Let $X \subset \mathbb{A}^n$ be an affine variety. Suppose that $f_1, f_2, \dots, f_k$ generate the ideal I of X. Then the tangent space of X at p, considered as a subspace of the tangent space to $\mathbb{A}^n$, via the inclusion of X in $\mathbb{A}^n$, is equal to the kernel of the Jacobian matrix.*

Proof. Clearly it is easier to give the dual description of the cotangent space.

If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,p}$ and $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X,p}$, then clearly the natural map $\mathfrak{m} \longrightarrow \mathfrak{n}$ is surjective, so that the induced map on contangent spaces is surjective. Dually, the induced map on the Zariski tangent space is injective, so that T_pX is indeed included in $T_p\mathbb{A}^n$.

We may as well choose coordinates $x_1, x_2, \dots, x_n$ so that p is the origin. In this case $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$ and $\mathfrak{n} = \mathfrak{m}/I$. Moreover $\mathfrak{m}/\mathfrak{m}^2$ is the vector space spanned by $dx_1, dx_2, \dots, dx_n$, where dx_i denotes the equivalence class $x_i + \mathfrak{m}^2$, and $\mathfrak{n}/\mathfrak{n}^2$ is canonically isomorphic to $\mathfrak{m}/(\mathfrak{m}^2 + I)$. Now the transpose of the Jacobian matrix, defines a linear map

$$K^k \longrightarrow K^n = T_p^*\mathbb{A}^n,$$

and it suffices to prove that the image of this map is the kernel of the map

$$df \colon \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2.$$

Let $g \in \mathfrak{m}$. Then

$$g(x) = \sum a_i x_i + h(x),$$

where $h(x) \in \mathfrak{m}^2$. Thus the image of $g(x)$ in $\mathfrak{m}/\mathfrak{m}^2$ is equal to $\sum_i a_i dx_i$. Moreover, by standard calculus a_i is nothing more than

$$a_i = \left.\frac{\partial g}{\partial x_i}\right|_p .$$

Thus the kernel of the map df is generated by the image of f_i in $\mathfrak{m}/\mathfrak{m}^2$, which is

$$\sum_j \left.\frac{\partial f_i}{\partial x_j}\right|_p dx_j,$$

which is nothing more than the image of the Jacobian. □

Lemma 3.4. *Let X be a quasi-projective variety. Then the function*

$$\lambda\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(x)$ is the dimension of the Zariski tangent space at x.

Proof. Clearly this result is local on X so that we may assume that X is affine. In this case the Jacobian matrix defines a morphism π from X to the space of matrices and the locus where the Zariski tangent space has a fixed dimension is equal to the locus where this morphism lands in the space of matrices of fixed rank. Put differently the function λ is the composition of π and an affine linear function of the rank on the space of matrices. Since the rank function is upper semicontinuous, the result follows. □

Lemma 3.5. *Every irreducible quasi-projective variety is birational to a hypersurface.*

Proof. Let X be a quasi-projective variety of dimension k, with function field L/K. Let $L/M/K$ be an intermediary field, such that M/K is purely transcendental of transcendence degree, so that L/M is algebraic. As L/M is a finitely generated extension, it follows that L/M is finite. Suppose that L/M is not separable. Then there is an element $y \in L$ such that $y \notin M$ but $x_1 = y^p \in M$. We may extend x_1 to a transcendence basis $x_1, x_2, \dots, x_k$ of M/K. Let M' be the intermediary field generated by $y, x_2, x_3, \dots, x_k$. Then M'/K is a purely transcendental extension of K and

$$[L : M] = [L : M'][M' : M] = p[L : M'].$$

Repeatedly replacing M by M' we may assume that L/M is a separable extension.

By the primitive element Theorem, L/M is generated by one element, say α. It follows that there is polynomial $f(x) \in M[x]$ such that α is a root of $f(x)$. If $M = K(x_1, x_2, \dots, x_k)$, then clearing denominators, we may assume that $f(x) \in K[x_1, x_2, \dots, x_k][x] \simeq K[x_1, x_2, \dots, x_{k+1}]$. But then X is birational to the hypersurface defined by $F(X)$, where $F(X)$ is the homogenisation of $f(x)$. □

Proposition 3.6. *The set of smooth points of any variety is Zariski dense.*

Proof. Since the dimension of the Zariski tangent space is upper semicontinuous, and always at least the dimension of the variety, it suffices to prove that every irreducible variety contains at least one smooth point. By (3.5) we may assume that X is a hypersurface. Passing to an affine open subset, we may assume that X is an affine hypersurface. Let f be a defininining equation, so that f is an irreducible polynomial. Then the set of singular points of X is equal to the locus of points where every partial derivative vanishes. If g is a non-zero partial derivative of f, then g is a non-zero polynomial of degree one less than f, and so cannot vanish on X.

If all the partial derivatives of f are the zero polynomial, then f is a pth power, where the characteristic is p, which contradicts the fact that f is irreducible. □

Note that if we take a smooth variety X and blow up a point p, then the exceptional divisor E is canonically the projectivisation of the Zariski tangent space to X at p,

$$E = \mathbb{P}(T_pX).$$

Indeed the point is that E picks up the different tangent directions to X at p, and this is exactly the set of lines in T_pX.

One defines the Zariski tangent space to a scheme X, at a point x, using exactly the same definition, the dual of

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring. However in general, if we have the equality of dimensions of both the Zariski tangent space and the local dimension, we only call X **regular at** $x \in X$. Smoothness is a more restricted notion in general.

Having said this, if X is a quasi-projective variety over an algebraically closed field then X is smooth as a variety if and only if it is smooth as a scheme over $\operatorname{Spec} k$. In fact an abstract variety over $\operatorname{Spec} k$ is smooth if and only if it is regular. Note that if x is a specialisation of ξ and X is regular at x then X is regular at ξ, so it is enough to check that X is regular at the closed points.

One can sometimes use the Zariski tangent space to identify embedded points. If X is a scheme and $Y = X_{\text{red}}$ is the reduced subscheme then $x \in X$ is an embedded point if

$$\dim T_xX > \dim T_xY,$$

and X is reduced away from x. For example, if X is not regular at x but Y is regular at x then $x \in X$ is an embedded point.

It is interesting to see which toric varieties are smooth. The question is local, so we might as well assume that $X = U_\sigma$ is affine. If $\sigma \subset N_{\mathbb{R}}$ does not span $N_{\mathbb{R}}$, then $X \simeq U_{\sigma'} \times \mathbb{G}_m^l$, where σ' is the same cone as

σ embedded in the space it spans. So we might as well assume that σ spans $N_{\mathbb{R}}$. In this case X contains a unique fixed point x_σ which is in the closure of every orbit. Since X only contains finitely many orbits, it follows that X is smooth if and only if X is regular at x_σ. The maximal ideal of x_σ is generated by χ^u, where $u \in S_\sigma$. The square of the maximal is generated by χ^{u+v}, where u and v are two elements of S_σ. So a basis for $\mathfrak{m}/\mathfrak{m}^2$ is given by elements of S_σ that are not sums of two elements. Since the elements of S_σ generate the group M, the elements of S_σ which are not sums of two elements, must generate the group. Given an extremal ray of $\check{\sigma}$, a primitive generator of this ray is not the sum of two elements in S_σ. So $\check{\sigma}$ must have n edges and they must generate M. So these elements are a basis of the lattice and in fact $X \simeq \mathbb{A}^n_k$.