23. Fano varieties

Definition 23.1. *A smooth projective variety is **Fano** if $-K_X$ is ample.*

Example 23.2. *Let $X \subset \mathbb{P}^r$ be a smooth hypersurface of degree d. By adjunction*

$$K_X = (K_{\mathbb{P}^r} + X)|_X = (d - r - 1)H,$$

where H is the class of a hyperplane. Thus X is Fano if and only if $d \leq r$.

Note that the product of Fano varieties is Fano. If C is a smooth projective curve then C is Fano if and only if $C \simeq \mathbb{P}^1$. A Fano surface is called a del Pezzo surface. What are the del Pezzo surfaces? $\mathbb{P}^2$ is a del Pezzo surface. $\mathbb{P}^1 \times \mathbb{P}^1$ is a del Pezzo surface; either use the fact that it is product of Fanos or use the fact that it is isomorphic to a quadric in $\mathbb{P}^3$. A smooth cubic surface is a del Pezzo surface. Let S be the blow up of $\mathbb{P}^2$ at a point. Then there is a morphism $S \longrightarrow \mathbb{P}^1$, with fibres copies of $\mathbb{P}^1$.

The degree d of a del Pezzo surface is the self-interesection of $-K_S$.

Theorem 23.3. *Let $\pi \colon S \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ along r general points.*

Then S is a del Pezzo surface if and only if $r \leq 8$.

Here general means that the points belong to a Zariski open subset of the product of r copies of $\mathbb{P}^2$.

Proof. Note that

$$d = K_S^2 = K_{\mathbb{P}^2}^2 - r = 9 - r.$$

If S is a del Pezzo surface then $d > 0$ and so $r \leq 8$.

Now suppose that $r \leq 8$. Suppose that we blow up $p_1, p_2, \dots, p_r$ with exceptional divisors $E_1, E_2, \dots, E_r$. Then

$$K_S = \pi^* K_{\mathbb{P}^2} + \sum E_i.$$

We have to show that $-K_S$ is ample. We will just prove a small part of this result. If $r \leq 6$ then we will show that $-K_S$ is very ample. We assume that no three points of $p_1, p_2, \dots, p_r$ are collinear and no six points lie on a conic. We have to show that the linear system $|-K_S|$ separates points and tangent vectors.

The linear system

$$|-K_{\mathbb{P}^2}| = |3L|,$$

is the linear system of all cubic curves. Thus divisors in the linear system

$$|-K_S| = |3\pi^* L - \sum E_i|,$$

correspond to cubics through $p_1, p_2, \dots, p_r$.

Suppose that we pick two points x and $y \in S$. This gives us two points p_{r+1} and p_{r+2} in $\mathbb{P}^2$. If x and y don't belong to the exceptional divisors, then we have $r + 2 \leq 8$ distinct points in $\mathbb{P}^2$. We check that these impose independent conditions on cubics. We have to check that for all $i \leq r + 1 \leq 7$ we can find a cubic through $p_1, p_2, \dots, p_i$ not containing p_{i+1}.

If $i \leq 6$ then pick three pairs of lines. As no three points of $p_1, p_2, \dots, p_r$ are collinear, then we can choose our lines not containing p_{i+1}. If $i = 7$ then pick a conic through five points and a line through the last point.

Thus if x and y don't lie on an exceptional divisor, then we may find a divisor in the linear system $|-K_S|$ through x not through y.

If x belongs to an exceptional divisor then $p_{r+1} = p_j$ for some j. Let's suppose $p_{r+1} = p_r$. Suppose that y does not belong to an exceptional divisor.

Then we need to find a cubic through $p_1, p_2, \dots, p_r$ with the tangent direction determined by x not passing through p_{r+2}.

If $r = 7$ then $|-K_S|$ defines a two to one map to $\mathbb{P}^2$, branched over a quartic curve C, $\pi \colon S \longrightarrow \mathbb{P}^2$. It follows, by Riemann-Hurwitz that

$$K_S = \pi^*(K_{\mathbb{P}^2} + 1/2C),$$

so that $-K_S$ is the pullback of an ample divisor L. Thus $-K_S$ is ample. In fact $-2K_S$ is very ample.

If $r = 8$ then $|-K_S|$ defines a pencil, $\pi \colon S \longrightarrow \mathbb{P}^1$. Two cubics intersect in 8 points, so that the pencil has a base point and π is not a morphism. $|-2K_S|$ defines a double cover of a quadric cone in $\mathbb{P}^3$. $-3K_S$ is very ample. □

It is interesting to consider the image of the anticanonical linear system. If $r = 6$ so that $d = 3$ then $|-K_S|$ is the linear system of cubics through six points. The linear system of all cubics is 9 dimensional and we have already seen that six points impose independent conditions. Therefore $|-K_S|$ is a three dimensional linear system, and S is embedded as a smooth hypersurface of degree 3 in $\mathbb{P}^3$, a cubic surface. This raises the interesting question, is every smooth cubic the blow up of $\mathbb{P}^2$ in six points?

Let's count moduli. The space of six points on $\mathbb{P}^2$ has dimension $6 \cdot 2 = 12$. The automorphism group of $\mathbb{P}^2$ is PGL(3) which has dimension

$3 \cdot 3 - 1 = 8$. Thus there is a four dimensional family of non-isomorphic del Pezzo surfaces obtained by blowing up 6 points.

The space of cubics is a linear system of dimension

$$\binom{3+3}{3} - 1 = \binom{6}{3} - 1 = 19.$$

The automorphism group of $\mathbb{P}^3$ is $\mathrm{PGL}(4)$ which has dimension $4 \cdot 4 - 1 = 15$. Thus there is a four dimensional family of non-isomorphic cubic surfaces.

This suggests that every cubic surface is the blow up of $\mathbb{P}^2$ at six points. Unfortunately, it is hard to prove this directly. The natural map

$$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^{19}$$

which sends a six-tuple of points $(p_1, p_2, p_3, p_4, p_5, p_6)$ to the corresponding cubic surface is a rational map, not a morphism.

If $r = 5$ then $d = 4$. The linear system of cubics through five points is a copy of $\mathbb{P}^4$. Thus we get a surface $S \subset \mathbb{P}^4$ of degree 4. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_S \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

If we twist this by $\mathcal{O}_{\mathbb{P}^4}(2)$, we get

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{O}_{\mathbb{P}^4}(2) \longrightarrow \mathcal{O}_S(2) \longrightarrow 0.$$

Taking global sections we get a left exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \longrightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \longrightarrow H^0(S, \mathcal{O}_S(2)).$$

Now

$$h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \binom{4+2}{2} = \binom{6}{2} = 15.$$

On the other hand the linear system $|\mathcal{O}_S(2))|$ corresponds to sextics in $\mathbb{P}^2$ which have multiplicity two at five points. It is 3 conditions to be singular at any given point (in local coordinates we need the constant term and the coefficient of both x and y to vanish). Thus

$$h^0(S, \mathcal{O}_S(2)) = \binom{6+2}{2} - 3 \cdot 5 = 13.$$

Thus

$$h^0(\mathbb{P}^4, \mathcal{I}_S(2) \geq 2.$$

It follows that there are two quadratic polynomials which vanish on S, that is, there are two quadric hypersurfaces Q_1 and Q_2 which contain S. Now the intersection of two quadrics is a surface of degree 4, so that we must have $S = Q_1 \cap Q_2$ (as schemes).

Conversely suppose we are given two quadrics Q_1 and Q_2 whose intersection is a smooth surface S.

$$K_{Q_1} = (K_{\mathbb{P}^4} + Q_1)|_{Q_1} = -3H.$$

Thus

$$K_S = (K_{Q_1} + S)|_S = -H,$$

so that S is anticanonically embedded.

Let's count moduli again. The moduli space of surfaces which are the blow up of five points has dimension 2. Picking Q_1 and Q_2 is like picking a pencil l of quadrics, that is, a line l in the space of all quadrics, that is a point l of a Grassmannian. The space of all quadrics is a copy of $\mathbb{P}^{14}$ and so we get a

$$2(14-1) = 26,$$

dimensional family. The automorphism group of $\mathbb{P}^4$ is PGL(5) which has dimension 24, so we get two dimensions of moduli.

If $r = 4$ then we blow up four points of $\mathbb{P}^2$. There is no moduli and our surface gets embedded as a degree five surface in $\mathbb{P}^5$. If $r \leq 3$ then S is toric, since three points in general points are the same as the three coordinate points.

Theorem 23.4. *Let S be a del Pezzo surface.*

Then S is isomorphic either to

(1) $\mathbb{P}^1 \times \mathbb{P}^1$, or
(2) $\mathbb{P}^2$ blown up in $r \leq 8$ points.

What can we say about lines on the cubic, from this point of view? Well, $C \subset S$ is a line if and only if

$$1 = H \cdot C = -K_S \cdot C.$$

Since a line is isomorphic to $\mathbb{P}^1$, we see that lines on the cubic surface are the same as -1-curves. What are the -1-curves on $\mathbb{P}^2$ blown up at $r \leq 8$ points?

We have

$$\operatorname{Pic}(S) = \mathbb{Z} \cdot H \bigoplus_{i=1}^{r} \mathbb{Z}E_i,$$

where $H = \pi^* L$ is the pullback of a line. Thus a general curve in S has class

$$aH - \sum a_i E_i,$$

where a, $a_1, a_2, \dots, a_r$ are integers. The class of K_S is

$$-3H + \sum E_i.$$

Note that

$$\begin{aligned} H^2 &= 1, \\ H \cdot E_i &= 0 \\ E_i \cdot E_j &= \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If C is a -1-curve then $K_S \cdot C = -1$ and $C^2 - 1$. This gives us two Diophantine equations:

$$-3a + \sum a_i = -1 \qquad \text{and} \qquad a^2 - \sum a_i^2 = -1.$$

Let us start by guessing some solutions. We already know that if we blow up a point then the exceptional divisor is a -1-curve. Thus $a = 0$,

$$a_i = \begin{cases} -1 & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$

is a solution, for any k. There are r such exceptional curves.

The self-intersection of a line is 1. If we blow up a point on the line then the self-intersection of the strict transform is 0. If we blow up two points then the self-intersection of the strict transform is -1. Thus the strict transform of a line through two points is a -1-curve. There are

$$\binom{r}{2},$$

such lines. These correspond to the solutions $a = 1$,

$$a_i = \begin{cases} 1 & \text{if } i = k \text{ or } i = l \\ 0 & \text{otherwise,} \end{cases}$$

for any pair $k \neq l$.

The self-intersection of a conic is 4. If we blow up a five point on the conic then the self-intersection of the strict transform is -1. Thus the strict transform of a conic through five points is a -1-curve. There are

$$\binom{r}{5},$$

such conics. These correspond to the solutions $a = 2$,

$$a_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise,} \end{cases}$$

for any set I of five indices.

If $r = 6$ this gives

$$6 + \binom{6}{2} + \binom{6}{5} = 6 + 15 + 6 = 27$$

-1-curves. As a cubic surface has no more than 27 lines, every smooth cubic surface contains 27 lines.

If $r = 5$ we get

$$5 + \binom{5}{2} + \binom{5}{5} = 5 + 10 + 1 = 16,$$

-1-curves. Thus there are 16 lines on the intersection of two quadrics in $\mathbb{P}^4$.

One can check that if $r \leq 7$ that there are no other solutions.

We end with one of the most impressive and important results in the classification of surfaces:

Theorem 23.5 (Castelnuovo)**.** *Let S be a smooth projective surface and let $C \subset S$ be a curve.*

Then C is a -1-curve if and only if there is birational morphism $\pi\colon S \longrightarrow T$, which blows up a smooth point $p \in T$, with exceptional divisor C.

Proof. One direction is clear; if π blows up a smooth point then E is a -1-curve.

We prove the converse. Let H be an ample divisor. Then $H \cdot E = k > 0$. Possibly replacing H by a multiple, we may assume that $k > 1$ and furthermore that $K_S + H$ is ample. Let

$$D = K_S + (k-1)E + H.$$

Then

$$D \cdot C = (K_S + (k-1)E + H) \cdot C = -1 - (k-1) + k = 0.$$

On the other hand, let m be a positive integer such that $A = m(K_S + H)$ is very ample. By Serre vanishing we may assume that

$$h^1(S, A) = 0.$$

Note that if $B \in |A|$ then $B + m(k-1)E \in |mD|$. Thus the linear system $|mD|$ separates points and tangent vectors outside of the support of E. In particular the base locus of $|mD|$ is contained in E.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Twisting by mD we get

$$0 \longrightarrow \mathcal{O}_S(mD - E) \longrightarrow \mathcal{O}_S(mD) \longrightarrow \mathcal{O}_E(mD) \longrightarrow 0.$$

Now

$$mD - E = A + lE,$$

where $l = m(k-1) - 1$. We show that

$$h^1(S, A + iE) = 0,$$

for $0 \leq i \leq l$. The case $i = 0$ follows by assumption. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(A + (i-1)E) \longrightarrow \mathcal{O}_S(A + iE \longrightarrow \mathcal{O}_E(A + iE) \longrightarrow 0.$$

$(A + iE) \cdot E = mk - i > 0$. Thus

$$h^1(E, A + iE) = h^1(\mathbb{P}^1, (mk - i)p) = 0.$$

Taking the long exact sequence of cohomology we see that

$$h^1(S, A + iE) = h^1(S, A + (i-1)E) = 0,$$

by induction on i. Thus the linear system $|mD|$ is base point free. Consider the image T of the corresponding morphism $S \longrightarrow \mathbb{P}^N$. Since $|mD|$ separates points and tangent vectors outside E, T is a surface and the corresponding morphism $\pi \colon S \longrightarrow T$ is birational. Let G be the restriction of a hyperplane from $\mathbb{P}^N$. Then $\pi^* G = mD$. Let C' be the image of C. Then

$$0 = (mD) \cdot C = \pi^* G \cdot C = G \cdot C'.$$

As G is ample, C' is zero dimensional and so C' is a point. Thus π contracts C.

It remains to prove that T is smooth. Consider the divisor $mD - E$.

$$E \cdot (mD - E) = 1.$$

By the same argument as above, $mD - E$ is base point free. In fact it separates points and tangent vectors, so that it is very ample. We may find $\Sigma \in |mD - E|$ such that Σ is smooth. $\Sigma \cdot C = 1$. Thus $\Sigma' = \pi(\Sigma)$ is smooth. But $\Sigma + C \in |mD|$ and $\pi(\Sigma + C) = \Sigma'$ so that Σ' is Cartier. But then T is smooth. $\square$