20. Cohomology of projective space

Let us calculate the cohomology of projective space.

Theorem 20.1. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.

(2)

$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)

$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$

is a perfect pairing of finitely generated free A-modules.

Proof. Let

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$

Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute sheaf cohomology using this cover. Now

$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$

where

$$x_I = \prod_{i \in I} x_i.$$

Thus Čech cohomology is the cohomology of the complex

$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$

The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map

$$\prod_{i} S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$

The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent for at least one i. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{ x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1}x_1^{-1}\dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0}x_1^{l_1}\dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0}x_1^{m_1}\dots x_r^{m_r} = x_0^{-1-l_0}x_1^{-1-l_1}\dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$, by induction, and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. □

Theorem 20.2 (Serre vanishing)**.** *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) $H^i(X,\mathcal{F})$ *are finitely generated* A*-modules.*
(2) *There is an integer* n_0 *such that* $H^i(X,\mathcal{F}(n)) = 0$ *for all* $n \geq n_0$ *and* $i > 0$*.*

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X,\mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (20.1). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X,\mathcal{F}) = 0,$$

if $i > r$, by Grothendieck's vanishing theorem. On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X,\mathcal{F}(n)) \simeq H^{i+1}(X,\mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. □

Theorem 20.3. *Let* A *be a Noetherian ring and let* X *be a proper scheme over* A*. Let* $\mathcal{L}$ *be an invertible sheaf on* X*. TFAE*

(1) $\mathcal{L}$ *is ample.*
(2) *For every coherent sheaf* $\mathcal{F}$ *on* X *there is an integer* n_0 *such that*

$$H^i(X,\mathcal{F}\otimes\mathcal{L}^n) = 0,$$

for $n \geq n_0$*.*

Proof. Suppose that (1) holds. Pick a positive integer m such that $\mathcal{M} = L^{\otimes m}$ is very ample. Let $\mathcal{F}_r = \mathcal{F}\otimes\mathcal{L}^r$, for $0 \leq r \leq m-1$. By (20.2) we may find n_r depending on r such that $H^i(X,\mathcal{F}_r\otimes\mathcal{M}^n) = 0$ for all $n \geq n_r$ and $i > 0$. Let p be the maximum of the n_r. Given $n \geq n_0 = pm$, we may write $n = qm + r$, for some $0 \leq r \leq m-1$ and $q \geq p$. Then

$$H^i(X,\mathcal{F}\otimes\mathcal{L}^n) = H^i(X,\mathcal{F}_r\otimes\mathcal{M}^q) = 0,$$

for any $i > 0$. Hence (1) implies (2).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \geq n_0$ there is an open neighbourhood U of p, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \geq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$