2. Cartier Divisors

We now turn to the notion of a Cartier divisor.

Definition 2.1. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the* ***total quotient ring*** *of A.*

Given a scheme X, let $\mathcal{K}$ be the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the ***sheaf of total quotient rings****.*

Definition 2.2. *A* ***Cartier divisor*** *on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}_X^*$.*

In other words, a Cartier divisor is specified by an open cover U_i and a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal.

Definition 2.3. *Let X be a scheme satisfying $(*)$. Then every Cartier divisor determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X is **factorial**. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Example 2.4. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. The line l, given by $x = z = 0$, is a Weil divisor which is not Cartier (one needs to check that the ideal $\langle x, z \rangle$ inside $\mathcal{O}_{Q,0}$ is not principal). The hyperplane $x = 0$ cuts out the double line $2l$.*

Definition-Lemma 2.5. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$, where multiplication corresponds to tensor product and the inverse to the dual.

Proof. It is clear that tensor product is commutative and associative and that $\mathcal{O}_X$ plays the role of the identity. But if $\mathcal{M} = \operatorname{Hom}(\mathcal{L}, \mathcal{O}_X)$ then

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{L} \simeq \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X. \qquad \square$$

Definition 2.6. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 2.7. *Let X be a scheme.*

(1) *The association $D \longrightarrow \mathcal{O}_X(D)$ defines a correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.*
(2) *If $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, as subsheaves of $\mathcal{K}$*
(3) *Two Cartier divisors D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ (not necessarily as subsheaves of $\mathcal{K}$.*

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. By (2.7) it suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)).$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

It follows that $I = A_\sigma \chi^u$, so that $D = (\chi^u)$ is principal. In particular, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 2.8. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. We have already seen* (1.13) *that the class group is $\mathbb{Z}_2$.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \mid \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v\rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by u. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X),$$

is an embedding. It is an interesting to compare $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ on a toric variety. Denote by $\mathrm{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 2.9. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}_T(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^s & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0
\end{array}$$

In particular

$$\rho(X) = \mathrm{rank}(\mathrm{Pic}(X)) \leq \mathrm{rank}(\mathrm{Cl}(X)) = s - n.$$

Further $\mathrm{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If $\mathcal{L}$ is an invertible sheaf then $\mathcal{L}|_U$ is trivial. Suppose that $\mathcal{L} = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier and exactness of the top row is easy.

Finally, $\mathrm{Pic}(X)$ is subgroup of the direct sum of $M/M(\sigma)$ and each of these is a lattice, whence $\mathrm{Pic}(X)$ is torsion free. $\square$