18. Cohomology of affine schemes

Proposition 18.1. *Let I be an injective module over a noetherian ring A.*

Then the sheaf $\tilde{I}$ on $X = \operatorname{Spec} A$ is flasque.

Corollary 18.2. *Let X be a noetherian scheme.*

Then ever quasi-coherent sheaf $\mathcal{F}$ on X can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.

Proof. Let $U_i = \operatorname{Spec} A_i$ be a finite open affine cover of X and let $\mathcal{F}|_{U_i} = \tilde{M}_i$ for each i. Pick an embedding of M_i into an injective A_i-module I_i. Let $f_i \colon U_i \longrightarrow X$ be the inclusion and let

$$\mathcal{G} = \bigoplus_i f_{i*}\tilde{I}_i.$$

Now for each i there is an injective map $\mathcal{F}|_{U_i} \longrightarrow \tilde{I}_i$, which induces a map $\mathcal{F} \longrightarrow f_{i*}\tilde{I}_i$. This induces a map $\mathcal{F} \longrightarrow \mathcal{G}$, which is clearly injective.

But $\tilde{I}_i$ is flasque and quasi-coherent on U_i, so that $f_{i*}\tilde{I}_i$ is flasque and quasi-coherent on X. But then $\mathcal{G}$ is flasque and quasi-coherent. □

Theorem 18.3 (Serre)**.** *Let X be a Noetherian scheme.*

TFAE

1. *X is affine,*
2. *$H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,*
3. *$H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.*

Proof. Suppose X is affine. Let $M = H^0(X, \mathcal{F})$ and take an injective resolution $I^\bullet$ of M in the category of A-modules. Then $\tilde{I}^\bullet$ is a flasque resolution of $\mathcal{F}$ on X. If we take global sections we get back the original injective resolution of M, so that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Thus (1) implies (2).

(2) implies (3) is immediate. So suppose that $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Fix a closed point p of X together with an open affine neighbourhood U of p and let $Y = X - U$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup \{p\}} \longrightarrow \mathcal{I}_Y \longrightarrow k(p) \longrightarrow 0.$$

This gives us an exact sequence

$$H^0(X, \mathcal{I}_Y) \longrightarrow H^0(X, k(p)) \longrightarrow H^1(X, \mathcal{I}_{Y \cup \{p\}}) \longrightarrow 0.$$

But then there is regular function $f \in H^0(X, \mathcal{O}_X)$ which is not zero at p and which does not vanish on U, so that $p \in X_f \subset U$ is an open neighbourhood of p. As $X_f = U_f$ (thinking of $f \in A = H^0(X, \mathcal{O}_X)$), it follows that X_f is affine.

As X is noetherian, it is compact, so that we can cover X by finitely many open affines, X_{f_i}, where $f_1, f_2, \dots, f_r \in A$.

Finally we check that $f_1, f_2, \dots, f_r$ generate the unit ideal. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

The last map α sends $(a_1, a_2, \dots, a_r)$ to $\sum a_i f_i$. It is surjective as it is surjective on stalks. $\mathcal{F}$ is then the kernel of α.

There is a filtration of $\mathcal{F}$ as follows:

$$\mathcal{F} \cap \mathcal{O}_X \subset \mathcal{F} \cap \mathcal{O}_X^2 \subset \mathcal{F} \cap \mathcal{O}_X^3 \subset \mathcal{F} \cap \mathcal{O}_X^r = \mathcal{F}.$$

The quotients are naturally $\mathcal{O}_X$-submodules of $\mathcal{O}_X$, that is, the quotients are coherent sheaves of ideals. Taking the long exact sequence of cohomology (r times), we get that $H^1(X, \mathcal{F}) = 0$. Taking the long exact sequence of cohomology of the sequence above, we get that α is surjective on global sections. But then

$$1 = \alpha(a_1, a_2, \dots, a_r) = \sum a_i f_i,$$

in the ideal generated by $f_1, f_2, \dots, f_r$. (II.2.17) shows that X is affine.

□