17. Higher vanishing

Theorem 17.1. *Let X be a noetherian topological space of dimension n.*

Then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$ and any sheaf of abelian groups.

The basic idea is to reduce to the case of the quotient of $\mathbb{Z}_U$. The first thing is to reduce to the finitely generated case. Recall that any ring is the direct limit of its finitely generated subrings, so really all we need is a couple of standard results about direct limits.

Suppose A is a directed set and $(\mathcal{F}_\alpha)$ is a direct system of sheaves indexed by A. Then we may take the direct limit $\varinjlim \mathcal{F}_\alpha$.

Lemma 17.2. *On a noetherian topological space, the direct limit of flasque is flasque.*

Proof. Suppose $(\mathcal{F}_\alpha)$ is a direct system of flasque sheaves. Suppose that $V \subset U$ are open subsets. For each i we have a surjection

$$\mathcal{F}_\alpha(U) \longrightarrow \mathcal{F}_\alpha(V).$$

Now $\varinjlim$ is an exact functor, so

$$\varinjlim \mathcal{F}_\alpha(U) \longrightarrow \varinjlim \mathcal{F}_\alpha(V),$$

is surjective. But on a noetherian topological space we have

$$(\varinjlim \mathcal{F}_\alpha)(U) = \varinjlim \mathcal{F}_\alpha(U),$$

and so

$$(\varinjlim \mathcal{F}_\alpha)(U) \longrightarrow (\varinjlim \mathcal{F}_\alpha)(V),$$

is surjective, so that $\varinjlim \mathcal{F}_\alpha$ is flasque. □

Proposition 17.3. *Let X be a noetherian topological space and let $(\mathcal{F}_\alpha)$ be a direct system of abelian sheaves indexed by A.*

Then there are natural isomorphisms

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha)$$

Proof. By definition of the limit, for each α, there are maps $\mathcal{F}_\alpha \longrightarrow \varinjlim \mathcal{F}_\alpha$. This gives a map on cohomology

$$H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Taking the limit of these maps gives a morphism

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

It is easy to check that this is an isomorphism for $i = 0$.

The general case follows using the notion of a δ-functor; see (III.2.9). □

Lemma 17.4. *Let $Y \subset X$ be a closed subset of a topological space, let $\mathcal{F}$ be a sheaf of abelian groups on Y and let $j\colon Y \hookrightarrow X$ be the natural inclusion.*

Then

$$H^i(Y, \mathcal{F}) \simeq H^i(X, j_*\mathcal{F}).$$

Proof. Suppose that $\mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$. Then $j_*\mathcal{I}^\bullet$ is a flasque resolution of $j_*\mathcal{F}$. □

It is customary to abuse notation and consider $\mathcal{F}$ as a sheaf on X, without bothering to write $j_*\mathcal{F}$.

Proof of (17.1). We introduce some convenient notation. Let $\mathcal{F}$ be a sheaf on X. If $Y \subset X$ is a closed subset then $\mathcal{F}_Y$ denotes the extension by zero of the sheaf $\mathcal{F}|_Y$; similarly if $U \subset X$ is an open subset, then $\mathcal{F}_U$ denotes the extension by zero of the sheaf $\mathcal{F}|_U$. Note that if $U = X - Y$ then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

The proof proceeds by Noetherian induction and induction on $n = \dim X$.

Step 1: We reduce to the case X is irreducible. We proceed by induction on the number m of irreducible components of X. If $m = 1$ there is nothing to prove. Otherwise let Y be an irreducible component of X and let $U = X - Y$. Let Z be the closure of U. Then $\mathcal{F}_U$ can be considered as a sheaf on Z, which has $m - 1$ irreducible components. By induction on m,

$$H^i(Y, \mathcal{F}|_Y) = H^i(Z, \mathcal{F}_U) = 0,$$

for all $i > n$, and so

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$, by considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

So we may assume that X is irreducible.

Step 2: Suppose that $n = 0$. Then X and the empty set are the only open subsets of X. In this case, to give a sheaf on X is the same as to give an abelian group, and it clear that taking global sections is an exact functor. But then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Thus we may assume that $n > 0$.

Step 3: Let

$$B = \bigcup_U \mathcal{F}(U),$$

and let A be the set of all finite subsets of B. Given $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the elements of α. Then A is a directed set and $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$. By (17.3) we may therefore assume that $\mathcal{F}$ is finitely generated.

Step 4: Suppose that $\beta \subset \alpha$ and let r be the cardinality of the difference. Then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_\beta \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is generated by r elements. So by induction on r and the long exact sequence of cohomology associated to the short exact sequence above, we are reduced to the case when $\mathcal{F}$ is generated by a single element, so that $\mathcal{F}$ is a quotient of $\mathbb{Z}_U$ for some open subset U,

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathbb{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

Step 5: We reduce to the case when $\mathcal{F} = \mathbb{Z}_U$.

For each $x \in U$, $\mathcal{R}_x \subset \mathbb{Z}_x = \mathbb{Z}$. If $\mathcal{R} = 0$ there is nothing to prove; otherwise let d be the smallest positive integer which appears in $\mathcal{R}_x$. Then

$$\mathcal{R}|_V = d \cdot \mathbb{Z}|_V,$$

for some non-empty open subset $V \subset U$. In this case, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Now $\mathcal{Q}$ is supported on a smaller set and so

$$H^i(X, \mathcal{Q}) = 0,$$

for all $i \geq n$. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{R}) = H^i(X, \mathbb{Z}_V),$$

for all $i > n$.

Thus we may assume that $\mathcal{F} = \mathbb{Z}_U$.

Step 6: Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_Y \longrightarrow 0,$$

where $Y = X - U$. By induction on the dimension,

$$H^i(X, \mathbb{Z}_Y) = 0,$$

for all $i \geq n$. Thus
$$H^i(X, \mathbb{Z}_U) = H^i(X, \mathbb{Z}),$$
for all $i > n$. But $\mathbb{Z}$ is a locally constant sheaf on an irreducible space, so that $\mathbb{Z}$ is flasque, and flasque is acyclic. $\square$