15. The canonical bundle and divisor

Definition 15.1. *Let X be a smooth variety of dimension n over a field k. The* ***canonical sheaf****, denoted ω_X, is the highest wedge of the sheaf of relative differentials,*

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

Note that ω_X is an invertible sheaf on X. We may write $\omega_X = \mathcal{O}_X(K_X)$, for some Cartier divisor K_X. The interesting thing is that we may generalise this:

Definition 15.2. *Let X be a normal variety over a field k. Let $U \subset X$ be the smooth locus, an open subset of X, whose complement has codimension at least two.*

The ***canonical divisor****, denoted K_X, is the Weil divisor obtained by picking a Weil divisor representing the invertible sheaf ω_U and then taking the closure.*

Note that the canonical divisor is only defined up to linear equivalence.

Definition 15.3. *Let X be a smooth projective variety over a field k. The* ***geometric genus*** *of X, denoted $p_g(X)$, is the dimension of the k-vector space $H^0(X, \omega_X)$. The m-th* ***plurigenus****, denoted $P_m(X)$, is the dimension of the k-vector space $H^0(X, \mathcal{O}_X(mK_X))$. The* ***irregularity*** *of X, denoted $q(X)$, is the dimension of the k-vector space $H^0(X, \Omega_{X/k})$.*

Note that $p_g(X) = P_1(X)$. If X is a curve, then $p_g(X) = P_1(X) = q(X)$.

Definition 15.4. *Let X be a scheme. We say that X satisfies* ***condition*** *S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 15.5 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

It is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 15.6. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the* *codimension of the complement is two. Let $f\colon U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Example 15.7. *Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of*

$$[S:T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A:B:C:D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed,

$$\frac{B^2}{A} = S^2T^2 = \frac{C^2}{D}.$$

is a regular function whose only pole is along $A = 0$ and $D = 0$, that is, only at $(0,0,0,0)$ of S.

Note that the coordinate ring

$$k[S^4, S^3T, ST^3, T^4] = \frac{k[A,B,C,D]}{\langle AD - BC, B^3 - A^2C, C^3 - BD^2\rangle},$$

is indeed not integrally closed in its field of fractions. Indeed,

$$\alpha = \frac{B^2}{A},$$

is a root of the monic polynomial $u^2 - BC$.

Theorem 15.8. *Let X and X' be two smooth projective varieties over a field k.*

If X and X' are birational then $p_g(X) = p_g(X')$, $P_n(X) = P_n(X')$ and $q(X) = q(X')$.

Proof. We will just prove that the geometric genus is a birational invariant. By symmetry, it suffices to show that $p_g(X') \leq p_g(X)$. By assumption there is a birational map $\phi\colon X \dashrightarrow X'$. Let $V \subset X$ be the largest open subset of X for which this map restricts to a morphism, $f\colon V \longrightarrow X'$. This induces a map of sheaves,

$$f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}.$$

Since these are both locally free of the same rank $n = \dim V$, taking the highest wedge, we get

$$f^*\omega_{X'} \longrightarrow \omega_V.$$

Since f is birational there is an open subset $U \subset V$ such that $f(U)$ is open in X' and f induces an isomorphism $U \longrightarrow f(U)$. Since a non-zero section of an invertible sheaf cannot vanish on any non-empty open subset, we have an injection on global sections

$$H^0(X', \omega_{X'}) \longrightarrow H^0(V, \omega_V).$$

So it suffices to show that the natural restriction map

$$H^0(X, \omega_X) \longrightarrow H^0(V, \omega_V),$$

is an isomorphism.

First off, we note that the codimension of the complement $X - V$ is at least two. Indeed, let P be a codimension one point. Then $\mathcal{O}_{X,P}$ is a DVR, as X is smooth. We already have a map of the generic point of X to X'. As X' is projective it is proper, so that there is a unique morphism $\operatorname{Spec} \mathcal{O}_{X,P} \longrightarrow X'$ compatible with ϕ. This morphism extends to a neighbourhood of P, so that f is defined in a neighbourhood of P, that is, $P \in V$.

To show that the restriction map is bijective, it suffices to show that if $U \subset X$ is an open subset for which $\omega_X|_U \simeq \mathcal{O}_U$, then the natural restriction map

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U \cap V, \mathcal{O}_{U \cap V})$$

is an isomorphism. But this follows as $U - V$ has codimension at least two and X is normal; any function on X which is regular in codimension two is regular. □

Definition 15.9. *Let Y be a smooth subvariety of a smooth variety X over a field k, with ideal sheaf $\mathcal{I}$. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf**. Its dual*

$$\mathcal{N}_{Y/X} = \mathbf{Hom}_{\mathcal{O}_Y}(\frac{\mathcal{I}}{\mathcal{I}^2}, \mathcal{O}_Y),$$

*is called the **normal sheaf** of Y in X.*

Note that by taking duals of the usual exact sequence on Y we get

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Theorem 15.10 (Adjunction formula)**.** *Let Y be a smooth subvariety of codimension r of a smooth variety X over a field k. Then*

$$\omega_Y \simeq \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}.$$

If $r = 1$ then if we consider Y as a divisor on X and put $\mathcal{L} = \mathcal{O}_X(Y)$, we get

$$\omega_Y \simeq \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y.$$

In terms of divisors,

$$K_Y = (K_X + Y)|_Y.$$

Proof. Follows from the exact sequence above, after taking highest wedge and then the dual. □

It is interesting to calculate the canonical divisor in the case of a smooth toric variety. To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if $z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor.

To calculate the zeroes and poles of this meromorphic differential, we may work locally about any invariant divisor. So we may assume that $X = U_\sigma$ is affine, isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m^{n-1}$. As usual, we reduce to the case when $n = 1$, in which case we have

$$\frac{\mathrm{d}z}{z},$$

which has a simple pole at 0.

Thus this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H.$$

One can check this with the formula one gets using the Euler sequence.