11. Blowing up schemes

We now consider how to define the blow up for an arbitrary scheme. Recall

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

Definition 11.1. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi\colon \mathbf{Proj}\,\mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 11.2. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \dots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \dots, x_n \rangle$,

$$Y = \mathbf{Proj}\,\mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \dots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

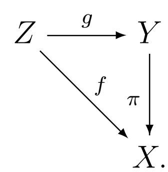
$$\langle y_i x_j - y_j x_i \rangle,$$

which are the usual equations of the blow up.

Definition 11.3. *Let $f\colon Y \longrightarrow X$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_X$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$.*

Theorem 11.4 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If $f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute



Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(1)$.

Now show that we are given $f\colon Z \longrightarrow X$. We first construct g, then show that if g is any factorisation of f, the pullback ideal sheaf is invertible and then finally show that g is unique.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of graded A-algebras

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow \bigoplus_{d=0}^{\infty} I^d,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_A,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(Z, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = g^{-1}(\mathcal{I} \cdot \mathcal{O}_Y) \cdot \mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1) \cdot \mathcal{O}_Z.$$

Thus $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf.

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

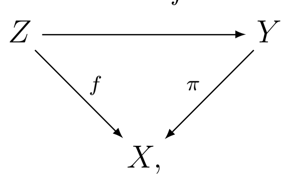
But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^*x_i$ and uniqueness follows. □

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 11.5. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is a coherent ideal sheaf $\mathcal{I}$ and a commutative diagram



where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.