10. Relative proj and projective bundles

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf $\mathcal{S}$ sheaf of graded $\mathcal{O}_X$-algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d,$$

where $\mathcal{S}_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. With a view towards what comes next, we denote global sections of $\mathcal{S}$ over U by $H^0(U, \mathcal{S})$. Then $\mathcal{S}(U) = H^0(U, \mathcal{S})$ is a graded A-algebra, and we get $\pi_U \colon \operatorname{Proj} \mathcal{S}(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{array}{ccc} \operatorname{Proj} \mathcal{S}(U_f) & \longrightarrow & \operatorname{Proj} \mathcal{S}(U) \\ \downarrow{\scriptstyle \pi_{U_f}} & & \downarrow{\scriptstyle \pi_U} \\ U_f & \longrightarrow & U. \end{array}$$

It is not hard to glue π_U together to get $\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative consruction has some similarities to the old construction.

Example 10.1. *If X is Noetherian and*

$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) *and* $\mathbf{Proj}\, \mathcal{S} = \mathbb{P}^n_X$.

Given a sheaf $\mathcal{S}$ satisfying (†), and an invertible sheaf $\mathcal{L}$, it is easy to construct a quasi-coherent sheaf $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$, which satisfies (†). The graded pieces of $\mathcal{S}'$ are $\mathcal{S}_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \mathbf{Proj}\, \mathcal{S}' \longrightarrow P = \mathbf{Proj}\, \mathcal{S},$$

which makes the diagram commute

$$\begin{array}{ccc} P' & \xrightarrow{\ \phi\ } & P \\ & {\scriptstyle \pi'} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & X, & \end{array}$$

and

$$\phi^* \mathcal{O}_P(1) \simeq \mathcal{O}_{P'}(1) \otimes \pi'^* \mathcal{L}.$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base. Even better π is projective if X has an ample line bundle; see (II.7.10).

There are two very interesting families of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf $\mathcal{E}$ of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\,\mathcal{S}$ is the **projective bundle** over X associated to $\mathcal{E}$. The fibres of $\pi\colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of $\mathbb{P}^n$, where $n = r - 1$. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is standard and the second statement reduces to the statement that the sections $x_0, x_1, \dots, x_n$ generate $\mathcal{O}_P(1)$.

The most interesting result is:

Proposition 10.2. *Let $g\colon Y \longrightarrow X$ be a morphism.*

Then a morphism $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjection $g^*\mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pulls back to a surjective morphism

$$g^*\mathcal{E} = f^*(\pi^*\mathcal{E}) \longrightarrow \mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf $\mathcal{L}$ and a surjective morphism of sheaves

$$g^*\mathcal{E} \longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniqueness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images $s_0, s_1, \ldots, s_n$ of the standard sections generate $\mathcal{L}$, and the result reduces to one we have already proved. $\square$