9. Enumerative geometry

Here is a typical question in enumerative geometry:

Question 9.1. *How many circles, in the usual real plane, pass through three points which are not collinear?*

The answer is one. Probably the easiest way to see this is to use synthetic geometry. Suppose the points are p, q and r. Let L and M be the bisectors of the two lines $\langle p, q\rangle$ and $\langle q, r\rangle$ (if S is a set of points then let $\langle S\rangle$ denote the span of S). Then it is easy to see that the point of intersection $L \cap M$ is the centre of the circle we are looking for and that this is the only circle through p, q and r.

However there are two entirely different ways to proceed, both of which will prove more fruitful, as they are more general.

Here is the first. Imagine moving the points around. Clearly the answer won't change (or better, if it did the original question does not really make sense). Now suppose that the points become collinear. In this case the only circle through these points is the straight line (a circle of infinite radius) containing them. Supposing that the answer does not change the answer must then be one in general. It is convenient to state more clearly the underlying assumption.

Principle 9.2. *(Principle of continuity) If we are given a problem in enumerative geometry, then the number of solutions is invariant under a continuous change of parameters.*

This is a very useful principle; unfortunately as stated it is clearly false, as there are some obvious counterexamples. The point is to change the definitions, so that this principle does indeed hold.

Question 9.3. *In how many points do two lines intersect?*

At first sight the answer would seem to be one; unfortunately some lines are parallel. In fact it is clear that the principle of continuity fails as well.

Example 9.4. *Let L be the line $y = 0$ and let M_t be the line $y = tx+1$, where $t \in K$. Then as t approaches zero, M approaches a line parallel to L, so that the number of points*

$$L \cap M_t,$$

is not constant, it is one for $t \neq 0$ but zero when $t = 0$.

Consider how the principle of continuity fails in this case. We have a sequence of points, $L \cap M_t$, without a limit. If we had a topological space (for example take $K = \mathbb{R}$), then this can only happen if the space is not compact. So we could fix the problem if we can compactify $\mathbb{A}^2$, by adding some points at infinity.

Definition 9.5. *Let K be a field and let V be a vector space V of dimension $n+1$. $\mathbb{P}(V)$ denotes the space of lines in V.* ***Projective space of dimension*** *n, denoted by $\mathbb{P}^n_K$, is the case $V = K^{n+1}$.*

Note that, as V is a vector space and not affine space, a line in V contains the origin.

Let us examine this definition more closely. Let V be a vector space of dimension $n+1$. Pick $v \in V - \{0\}$. Then v determines a line $\langle v \rangle$, in the usual way. On the other hand, if w is another non-zero vector, proportional to v, that is, $w = \lambda v$, for some $\lambda \neq 0 \in K$, then $\langle v \rangle = \langle w \rangle$. Thus we have proved:

Definition-Lemma 9.6. *Let V be a vector space. $\mathbb{P}(V)$ is equal to the set of points V, modulo the equivalence relation $\sim$, defined as $v \sim w$ iff $v = \lambda w$, $\lambda \in K^*$.*

The equivalence class of the vector v is denoted $[v]$.

Let us see what happens for small values of n. If $n+1=0$, then V does not contain any non-zero vectors, and so $\mathbb{P}^{-1}$ is empty. If $n+1=1$, then V contains a unique line and so $\mathbb{P}^0$ is a point.

The first interesting case is $\mathbb{P}^1$. Let $V = K^2$. Then $\mathbb{P}^1$ is the set of lines in the plane K^2. Suppose that $v = (X, Y) \in K^2 - \{0\}$. We denote the corresponding point of $\mathbb{P}^1$, by $[v] = [X : Y]$. Then the line spanned by v has a slope, provided $X \neq 0$, and this uniquely determines the line.

The slope $m = Y/X$ takes any value in K. Thus $\mathbb{A}^1 \subset \mathbb{P}^1$. On the other hand, we are only missing one point, corresponding to the line with slope infinity. Thus $\mathbb{P}^1 = \mathbb{A}^1 \cup \{p\}$, and we have compactified $\mathbb{A}^1$, by adding a single point. In fact, we sometimes refer to p as the point at infinity and even denote it by ∞ (the value of Y/X as it were). As an equivalence class, $p = [0 : 1]$.

Note that this situation is completely symmetric. Instead of looking at $y = Y/X$, we could consider $x = X/Y$. In this case we compactify $\mathbb{A}^1$, with coordinate x, by adding the point $q = [1 : 0]$.

It is useful to introduce some more notation to handle this. We denote by U_0 the locus of points of $\mathbb{P}^1$ where $X \neq 0$. As we have already seen, U_0 is a copy of $\mathbb{A}^1$. In this case $\mathbb{P}^1 = U_0 \cup \{[0 : 1]\}$.

Similarly we denote by U_1 the locus of points where $Y \neq 0$. Thus $\mathbb{P}^1 = U_1 \cup \{[1 : 0]\}$. The two sets U_0 and U_1 obviously intersect, along the locus $XY \neq 0$.

Let us see what happens for $\mathbb{P}^2$. Introduce coordinates (X, Y, Z) on $V \simeq K^3$. There are three obvious loci to consider, $X \neq 0$, $Y \neq 0$ and $Z \neq 0$. These induce three subsets of $\mathbb{P}^2$, U_0 and U_1 and U_2. I claim that that U_i is a copy of $\mathbb{A}^2$.

It is easy to see this algebraically. If $(X, Y, Z) \in K^3$ and $X \neq 0$, then $[X : Y : Z] = [1 : Y/X : Z/X]$. Thus the ratios $y = Y/X$ and $z = Z/X$ define coordinates on U_0 and identify U_0 with $\mathbb{A}^2$.

One can also see this geometrically. Any line through the origin of K^3 is determined by its intersection with the locus $X = 1$ (assuming it does intersect, that is, assuming the line lies in U_0). But the locus $X = 1$ is surely a copy of $\mathbb{A}^2$.

What is missing? In other words, what is $\mathbb{P}^2 - U_0$? This is the set of points with zero first coordinate, in other words all points of the form $[0 : Y : Z]$. But this is surely a copy of $\mathbb{P}^1$.

In other words we can compactify $\mathbb{A}^2$ by adding a copy of $\mathbb{P}^1$, to get $\mathbb{P}^2$. This copy of $\mathbb{P}^1$ is sometimes called the *line at infinity*.

As before, the situation is completely symmetric. Moreover, all of this generalises in an obvious fashion.

Definition 9.7. *Pick coordinates $X_0, X_1, \dots, X_n$ on K^{n+1}. We will refer to $[X_0 : X_1 : \dots : X_n]$ as **homogeneous coordinates** on $\mathbb{P}^n$. The subsets $U_0, U_1, \dots, U_n$, given as $X_i \neq 0$, which are copies of $\mathbb{A}^n$, are called the **standard open affine subsets**. Indeed the ratios $x_i = \frac{X_j}{X_i}$ define coordinates $x_1, x_2, \dots, \hat{x}_i, \dots, x_n$ on U_i.*

*The locus $X_i = 0$ is called the **hyperplane at infinity**. $\mathbb{P}^n = U_i \cup \{X_i = 0\}$.*

Note that what is at infinity, depends on our point of view. Note also that the term homogeneous coordinates is a bit of a misnomer. In fact $X_0, X_1, \dots, X_n$ are not functions at all, since they are not invariant under rescaling. The only thing that does make sense, is to ask where they are zero (which is invariant under rescaling).

Definition 9.8. *A subset Λ of a projective space $\mathbb{P}(V)$ is called **linear** if it is given as $\mathbb{P}(W)$, where $W \subset V$ is a linear subspace. The **dimension** of Λ is the dimension of W minus one.*

In other words a line l in $\mathbb{P}^2$ is the same as a plane W in the corresponding three dimensional vector space K^3.

Lemma 9.9. *Let Λ_1 and Λ_2 be two linear subspaces of $\mathbb{P}^n$ of dimension r and s.*

Then the dimension of the intersection is at least $r + s - n$.

Proof. Let W_1 and W_2 be the corresponding linear subspaces of V, where $\mathbb{P}^n = \mathbb{P}(V)$. Then W_1 has dimension $r+1$, W_2 has dimension $s+1$ and V has dimension $n+1$.

Clearly $\Lambda_1 \cap \Lambda_2 = \mathbb{P}(W_1 \cap W_2)$. On the other hand

$$\begin{aligned}\dim(W_1 \cap W_2) &\geq (r+1) + (s+1) - (n+1) \\ &= r + s - n + 1.\end{aligned}$$

□

The following example shows that we have fixed out problem concerning parallel lines.

Example 9.10. *Let l_1 and l_2 be two lines in $\mathbb{P}^2$. Then $l_1 \cap l_2$ intersect. Indeed the dimension of the intersection is at least zero ($= 1 + 1 - 2$) and the empty set has dimension -1.*

We will see later what happens when we take two parallel lines in $\mathbb{A}^2$ and compactify to $\mathbb{P}^2$. In practice it is often more efficient to work with the codimension and not the dimension.

Definition 9.11. *Let $\Lambda \subset \mathbb{P}^n$ be linear subspace. The **codimension** of Λ is equal to the difference $n - d$, where d is the dimension of Λ.*

The following is a simple restatement of (9.9); its virtue lies in the fact that is easier to remember and apply:

Lemma 9.12. *Let Λ_1 and Λ_2 be two linear subspaces of $\mathbb{P}^n$ of codimension r and s.*

Then the codimension of the intersection is at most $r+s$. That is the codimension of the intersection is at most the sum of the codimensions.

Let us go back to the principle of continuity. Unfortunately there is another problem.

Question 9.13. *In how many points do a line and a circle meet?*

Example 9.14. *Let L be the line $x = \sqrt{2}$ and C the circle $x^2 + y^2 = 1$ in $\mathbb{A}^2_{\mathbb{R}}$. Then L and C don't intersect.*

Now consider the family of lines L_t, $x = t$. Then $L_t \cap C$ depends on $t \in \mathbb{R}$. If $|t| < 1$ we get two points, if $t = \pm 1$, we get one and if $|t| > 1$ none at all. Thus the principle of continuity does not hold up.

The important thing to realise is that the problem here has nothing to do with the points of intersection moving off to infinity. The problem is that $\mathbb{R}$ is not algebraically closed, so that the equation $y^2 = -1$ has no solutions.

The solution is simple, we should replace $\mathbb{R}$ with $\mathbb{C}$. Now we always get two points (ignoring the possibility that $t = \pm 1$, which we will come back to), $x = t$, $y = \pm\sqrt{1 - t^2}$.

In practice, when we are considering problems in enumerative geometry, we will work almost exclusively over an algebraically closed field of characteristic zero, which for all intents and purposes means we work over $\mathbb{C}$. In fact working with other fields normally poses extra technical problems, so that working over $\mathbb{C}$ is the most convenient.