8. Affine Schemes II

As pointed out in §7, we need a slightly more general notion of a function than the one given above:

Definition 8.1. *Let R be a ring. We define a sheaf of rings $\mathcal{O}_X$ on the spectrum X of R as follows. Let U be any open set of X. A section $s \in \mathcal{O}_X(U)$ is by definition any function*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} R_{\mathfrak{p}},$$

where $s(\mathfrak{p}) \in R_{\mathfrak{p}}$, which is locally represented by a quotient. More precisely, given a point $\mathfrak{q} \in U$, there is an element $f \in R$ such that $\mathfrak{q} \in U_f \subset U$ and such that the section $s|_{U_f}$ is represented by a/f^n, for some $a \in R$ and $n \in \mathbb{N}$.

An ***affine scheme*** *is then any locally ringed space isomorphic to the spectrum of a ring with its associated sheaf. A* ***scheme*** *is a locally ringed space, which is locally isomorphic, as locally ringed space, to an affine scheme.*

It is not hard to see that $\mathcal{O}_X(U)$ is a ring (sums and products are defined in the obvious way) and that we do in fact have a sheaf rather than just a presheaf.

The key result is the following:

Lemma 8.2. *Let X be an affine scheme, isomorphic to the the spectrum of R and let $f \in R$.*

1. *For any $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to the local ring $R_{\mathfrak{p}}$.*
2. *The ring $\mathcal{O}_X(U_f)$ is isomorphic to R_f.*

In particular $\mathcal{O}_X(X) \simeq R$.

Proof. We first prove (1). There is an obvious ring homomorphism

$$\mathcal{O}_{X,\mathfrak{p}} \longrightarrow R_{\mathfrak{p}},$$

which just sends a germ (g, U) to its value $g(\mathfrak{p})$ at $\mathfrak{p}$.

On the other hand, there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which sends an element $r \in R$ to the pair (r, X). Suppose that $f \notin \mathfrak{p}$. Then $(1/f, U_f)$ defines an element of $\mathcal{O}_{X,\mathfrak{p}}$, and this element is an inverse of (f, X). It follows, by the universal property of the localisation, that there is a ring homomorphism,

$$R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{X,\mathfrak{p}},$$

which is the inverse map. Hence (1).

Now we turn to the proof of (2). As before there is an obvious ring homomorphism,

$$R \longrightarrow \mathcal{O}_X(U_f),$$

which induces a ring homomorphism

$$R_f \longrightarrow \mathcal{O}_X(U_f).$$

We have to show that this map is an isomorphism. We first consider injectivity. Suppose that $a/f^n \in R_f$ is sent to zero. Then for every $\mathfrak{p} \in \operatorname{Spec} R$, $f \notin \mathfrak{p}$, the image of a/f^n is equal to zero in $R_{\mathfrak{p}}$. For each such prime $\mathfrak{p}$ there is an element $h \notin \mathfrak{p}$ such that $ha = 0$ in R. Let $\mathfrak{a}$ be the annihilator of a in R. Then $h \in \mathfrak{a}$ and $h \notin \mathfrak{p}$, so that $\mathfrak{a}$ is not a subset of $\mathfrak{p}$. Since this holds for every $\mathfrak{p} \in U_f$, it follows that $V(\mathfrak{a}) \cap U_f = \emptyset$. But then $f \in \sqrt{\mathfrak{a}}$ so that $f^l \in \mathfrak{a}$, for some l. It follows that $f^l a = 0$, so that a/f^n is zero in R_f. Thus the map is injective.

Now consider surjectivity. Pick $s \in \mathcal{O}_X(U_f)$. By assumption, we may cover U_f by open sets V_i such that s is represented by $a_i/g_i^{n_i}$ on V_i. Replacing g_i by $g_i^{n_i}$ we may assume that $n_i = 1$. By definition $g_i \notin \mathfrak{p}$, for every $\mathfrak{p} \in V_i$, so that $V_i \subset U_{g_i}$. Now since sets of the form U_h form a base for the topology, we may assume that $V_i = U_{h_i}$. As $U_{h_i} \subset U_{g_i}$ it follows that $V(g_i) \subset V(h_i)$ so that

$$\sqrt{\langle h_i \rangle} \subset \sqrt{\langle g_i \rangle}.$$

But then $h_i^{n_i} \in \langle g_i \rangle$, so that $h_i^{n_i} = c_i g_i$. In particular

$$\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^{n_i}}.$$

Replacing h_i by $h_i^{n_i}$ and a_i by $c_i a_i$, we may assume that U_f is covered by U_{h_i}, and that s is represented by a_i/h_i on U_{h_i}.

We have already shown that $f^n = \sum b_i h_i$, where $b_1, b_2, \dots, b_k \in R$ and U_f can be covered by finitely many of the sets U_{h_i}. Thus we may assume that we have only finitely many h_i. Now on $U_{h_i h_j} = U_{h_i} \cap U_{h_j}$, there are two ways to represent s, one way by a_i/h_i and the other by a_j/h_j. By injectivity, we have $a_i/h_i = a_j/h_j$ in $R_{h_i h_j}$ so that for some n,

$$(h_i h_j)^n (h_j a_i - h_i a_j) = 0.$$

Since there are only finitely many i and j, we may assume that n is independent of i and j. We may rewrite this equation as

$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$

If we replace h_i by h_i^{n+1} and a_i by $h_i^n a_i$, then s is still represented by a_i/h_i and moreover

$$h_j a_i = h_i a_j.$$

Let $a = \sum_i b_i a_i$, where $f^n = \sum_i b_i h_i$. Then for each j,

$$\begin{aligned} h_j a &= \sum_i b_i a_i h_j \\ &= \sum_i b_i h_i a_j \\ &= f^n a_j. \end{aligned}$$

But then $a/f^n = a_j/h_j$ on U_{h_j}. But then a/f^n represents s on the whole of U_f. □

Note that by (2) of (8.2), we have achieved our aim of constructing a topological space from an arbitrary ring R, which realises R as a natural subset of the continuous functions.

Definition 8.3. *A morphism of schemes is simply a morphism between two locally ringed spaces which are schemes.*

The gives us a category, the category of schemes. Note that the category of schemes contains the category of affine schemes as a full subcategory and that the category of schemes is a full subcategory of the category of locally ringed spaces.

Theorem 8.4. *There is an equivalence of categories between the category of affine schemes and the category of commutative rings with unity.*

Proof. Let F be the functor that associates to an affine scheme, the global sections of the structure sheaf. Given a morphism

$$(f, f^{\#})\colon (X, \mathcal{O}_X) = \operatorname{Spec} B \longrightarrow (Y, \mathcal{O}_Y) = \operatorname{Spec} A,$$

of locally ringed spaces then let

$$\phi\colon A \longrightarrow B,$$

be the induced map on global sections. It is clear that F is then a contravariant functor and F is essentially surjective by (8.2).

Now suppose that $\phi\colon A \longrightarrow B$ is a ring homomorphism. We are going to construct a morphism

$$(f, f^{\#})\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y),$$

of locally ringed spaces. Suppose that we are given $\mathfrak{p} \in X$. Then $\mathfrak{p}$ is a prime ideal of B. But then $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is a prime ideal of A. Thus we get a function $f : X \longrightarrow Y$. Now if $\mathfrak{a}$ is an ideal of A, then

$f^{-1}(V(\mathfrak{a})) = V(\langle \phi(\mathfrak{a}) \rangle)$, so that f is certainly continuous. For each prime ideal $\mathfrak{p}$ of B, there is an induced morphism

$$\phi_{\mathfrak{p}} \colon A_{\phi^{-1}(\mathfrak{p})} \longrightarrow B_{\mathfrak{p}},$$

of local rings. Now suppose that $V \subset Y$ is an open set. We want to define a ring homomorphism

$$f^{\#}(V) \colon \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

Suppose first that $V = U_g$, where $g \in A$. Then $\mathcal{O}_Y(V) = A_g$ and $f^{-1}(V) \subset U_{\phi(g)}$. But then there is a restriction map

$$\mathcal{O}_X(U_{\phi(g)}) \simeq B_{\phi(g)} \longrightarrow \mathcal{O}_X(f^{-1}(V)).$$

On the other hand, composing there is a ring homomorphism

$$A \longrightarrow B_{\phi(g)}.$$

Since the image of g is invertible, by the universal property of the localisation, there is an induced ring homomorphism

$$A_g \longrightarrow B_{\phi(g)}.$$

Putting all of this together, we have defined $f^{\#}(V)$ when $V = U_g$. Since the sets U_g form a base for the topology, and these maps are compatible in the obvious sense, this defines a morphism

$$f^{\#} \colon \mathcal{O}_Y \longrightarrow f_* \mathcal{O}_X,$$

of sheaves. Clearly the induced map on local rings is given by $\phi_{\mathfrak{p}}$, and so $(f, f^{\#})$ is a morphism of local rings.

Finally it suffices to prove that these two assignments are inverse. The composition one way is clear. If we start with ϕ and construct $(f, f^{\#})$ then we get back ϕ on global sections. Conversely suppose that we start with $(f, f^{\#})$, and let ϕ be the map on global sections. Given $\mathfrak{p} \in X$, we get a morphism of local rings on stalks, which is compatible with ϕ and localisation, so that we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow{f^{\#}_{\mathfrak{p}}} & B_{\mathfrak{p}}. \end{array}$$

Let's compare $f(\mathfrak{p})$ and $\phi^{-1}(\mathfrak{p})$. If $r \notin f(\mathfrak{p})$ then the image of r in $A_{f(\mathfrak{p})}$ is a unit, so that $f^{\#}_{\mathfrak{p}}(r)$ is a unit. Hence $\phi(r) \notin \mathfrak{p}$, that is, $r \notin \phi^{-1}(\mathfrak{p})$. On the other hand, as $f^{\#}_{\mathfrak{p}}$ is a local ring homomorphism, it follows that the inverse image of a unit in $B_{\mathfrak{p}}$ is a unit in $A_{f(\mathfrak{p})}$. Pick $r \notin \phi^{-1}(\mathfrak{p})$. Then $\phi(r) \notin \mathfrak{p}$, and this is sent to a unit in $B_{\mathfrak{p}}$. Thus the image of r in $A_{f(\mathfrak{p})}$ is a unit and so $r \notin f(\mathfrak{p})$. Thus $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$.

Now let's compare $f^\#$ and the map $g^\#$ associated to ϕ. Their difference is a morphisms of sheaves,

$$f^\# - g^\# \colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X,$$

This morphism is zero on stalks, as we have seen, so that it is the zero morphism. Thus $f^\# = g^\#$. $\square$