7. Affine schemes I

Schemes were introduced by Grothendieck more than fifty years ago into the world of algebraic geometry. It might help to quickly review the reasons why schemes were introduced in the first place. Then in the course of these lectures we will see how the theory of schemes deals with the limitations of working with varieties.

Geometrically there are two compelling reasons to work with more general objects than varieties. Firstly, it is desirable to have a definition of an affine variety which is independent of any embedding into affine space. Compare this with the definition of a group. Originally groups were thought of as subsets of the set of permutations of a set, which are closed under composition and inverses. It is clearly much better to have the abstract definition of a group and then consider all the possible ways of embedding the group into permutation groups. This way one can think about groups being isomorphic, without worrying about a particular embedding. Similarly one of the big conceptual advances of the twentieth century was a definition of an abstract manifold.

Secondly if one looks at even the simplest families of varieties, some fibres (or members of the family) are not really varieties. For example, consider $S = V(xy - t) \subset \mathbb{A}^3$, a surface in $\mathbb{A}^3$. Projection down to the t-axis,

$$\pi \colon S \longrightarrow \mathbb{A}^1,$$

realises this surface as a family of curves in $\mathbb{A}^2$. If $a \neq 0$ then $C = V(xy - a) \subset \mathbb{A}^2$ is a hyperbola but if $a = 0$ then $C = V(xy) \subset \mathbb{A}^2$ is a pair of lines. This example is okay (but only because we allow reducible varieties). Now consider the $S = V(x^2 - ty) \subset \mathbb{A}^3$, a surface in $\mathbb{A}^3$. Projection down to the t-axis,

$$\pi \colon S \longrightarrow \mathbb{A}^1,$$

realises this surface as a family of curves in $\mathbb{A}^2$. If $a \neq 0$ then $C = V(x^2 - ay) \subset \mathbb{A}^2$ is a parabola but if $a = 0$ then $C = V(x^2) \subset \mathbb{A}^2$ is a line. But something is wrong here; a line is not really a conic, it is defined by a linear polynomial not a quadratic polynomial. We want something geometric corresponding to a doubled line.

Moreover there are other equally compelling reasons to enlarge the category of varieties, coming from other areas of mathematics. Suppose that we want to understand the polynomial equation

$$x^n + y^n = z^n.$$

In terms of arithmetic, we are interested in those 3-tuples (x, y, z), where x, y and $z \in \mathbb{Z}$. It is well known that determining the integral solutions is very hard, and it is natural to attack such problems by considering what happens over $\mathbb{C}$ and also what happens when we reduce modulo p, which are both considerably easier and shed light on what happens over the integers. In these terms, it seems that we have a single object X (determined by the equation) and we seek to understand X, by computing what happens when we look at the set

$$X(R) = \{ (x, y, z) \in R^3 \mid x^n + y^n = z^n \},$$

where R is a commutative ring. Note also in this context, that even over a field K, it is not enough to work with zero sets over the field. For example consider the field $\mathbb{R}$. Then the family of curves

$$x^2 + y^2 = t,$$

inside $\mathbb{R}^2$, where $t \in \mathbb{R}$, is not well behaved. For $t > 0$, we get a circle, for $t = 0$ we get a single point and for $t < 0$, we get the empty set. In other words, if we have an algebraic variety, it is not enough to consider the ordinary points over $\mathbb{R}$. This becomes even clearer if we work over a finite field. It is clear that different geometric objects, which have very different dimensions, will have the same zero set.

Finally, it is often useful to attack problems in commutative algebra, by considering the corresponding affine variety. In these terms, restricting to finitedly generated algebras without nilpotents is unnecessarily restrictive.

The definition of an affine scheme is motivated by the correspondence between affine varieties and finitely generated algebras over a field, without nilpotents. The idea is that we should be able to associate to any ring R, a topological space X, and a set of continuous functions on X, which is equal to R. In practice this is too much to expect and we need to work with a slightly more general object than a continuous function.

Now if X is an affine variety, the points of X are in correspondence with the maximal ideals of the coordinate ring $A = A(X)$. Unfortunately if we have two arbitrary rings R and S, then the inverse image of a maximal ideal won't be maximal. However it is easy to see that the inverse image of a prime ideal is a prime ideal.

Definition 7.1. *Let R be a ring. $X = \operatorname{Spec} R$ denotes the set of prime ideals of R. X is called the **spectrum** of R.*

Note that given an element of R, we may think of it as a function on X, by considering it value in the quotient. It is interesting to see what these functions look like in specific cases.

Example 7.2. *Suppose that we take $X = \operatorname{Spec} k[x,y]$. Now any element $f = f(x,y) \in k[x,y]$ defines a function on X. Suppose that we consider a maximal ideal of the form $\mathfrak{p} = \langle x-a, x-b\rangle$. Then the value of f at $\mathfrak{p}$ is equal to the class of f inside the quotient*

$$R/\mathfrak{p} = \frac{k[x,y]}{\langle x-a, x-b\rangle}.$$

If we identify the quotient with k, under the obvious identification, then this is the same as evaluating f at (a,b).

Example 7.3. *Now consider $\mathbb{Z}$. Suppose that we choose an element $n \in \mathbb{Z}$. Then the value of n at the prime ideal $\mathfrak{p} = \langle p\rangle$ is equal to the value of n modulo p. For example, consider $n = 60$. Then the value of this function at the point 7 is equal to $60 \mod 7 = 4 \mod 7$. Moroever 60 has zeroes at 2, 3 and 5, where both 3 and 5 are ordinary zeroes, but 2 is a double zero.*

Example 7.4. *Suppose that we take the ring $R = k[x]/\langle x^2\rangle$. Then the spectrum contains only one element, the prime ideal $\langle x\rangle$. Consider the element $x \in R$. Then x is zero on the unique element of the spectrum, but it is not the zero element of the ring.*

Now we wish to define a topology on the spectrum of a ring. We want to make the functions above continuous. So given an element $f \in R$, we want the set

$$\{\, \mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) = 0 \,\} = \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \langle f\rangle \subset \mathfrak{p} \,\},$$

to be closed. Given that any ideal $\mathfrak{a}$ is the union of all the principal ideals contained in it, so that the set of prime ideals which contain $\mathfrak{a}$ is equal to the intersection of prime ideals which contain every principal ideal contained in $\mathfrak{a}$ and given that the intersection of closed sets is closed, we have an obvious candidate for the closed sets:

Definition 7.5. *The* ***Zariski topology*** *on X is given by taking the closed sets to be*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subset \mathfrak{p}\,\},$$

where $\mathfrak{a}$ is any ideal of R.

Lemma 7.6. *Let R be a ring.*

Then $X = \operatorname{Spec} R$ is a topological space. Moreover the open sets

$$U_f = \{\, \mathfrak{p} \in R \mid f \notin \mathfrak{p} \,\},$$

form a base for the topology.

Proof. Easy check. □

By what we said above, the Zariski topology is the weakest topology so that the zero sets of $f \in R$ are closed.

Example 7.7. *Let k be a field. Then $\operatorname{Spec} k$ consists of a single point.*

Example 7.8. *Now consider $\operatorname{Spec} k[x]$. If k is an algebraically closed field, then by the Nullstellensatz, the maximal ideals are in correspondence with the points of k. However, since $k[x]$ is an integral domain, the zero ideal is a prime ideal. Since $k[x]$ is a PID, the proper closed sets of X consist of finite unions of maximal ideals. The closure of the point $\xi = \langle 0 \rangle$ is then the whole of X. In particular, not only is the Zariski topology, for schemes, not Hausdorff or T_2, it is not even T_1.*

Example 7.9. *Now consider $k[x, y]$, where k is an algebraically closed field. Prime ideals come in three types. The maximal ideals correspond to points of k^2. The zero ideal, whose closure consists of the whole of X. But there are also the prime ideals which correspond to prime elements $f \in k[x, y]$. The zero locus of f is then an irreducible curve C, and in fact the closure of the point $\xi = \langle f \rangle$ is the curve C. The proper closed sets thus consist of a finite union of maximal ideals, union infinite sets of the maximal ideals which consist of all points belonging to an affine curve C, together with the ideal of each such curve.*

Example 7.10. *Now suppose that k is not algebraically closed. For example, consider $\operatorname{Spec} \mathbb{R}[x]$. As before the closure of the zero ideal consists of the whole of X. The maximal ideals come in two flavours. First there are the ideals $\langle x - a \rangle$, where $a \in \mathbb{R}$. But in addition there are also the ideals*

$$\langle x^2 + ax + b = (x - \alpha - i\beta)(x - \alpha + i\beta) \rangle,$$

where a, b, α and $\beta > 0$ are real numbers, so that $b^2 - 4a < 0$.

Example 7.11. *There is a very similar (but more complicated) picture inside $\operatorname{Spec} \mathbb{R}[x, y]$. The set $V(x^2+y^2 = -1)$ does not contain any ideals of the first kind, but it contains many ideals of the second kind. In the classical picture, the conic does $x^2 + y^2 = -1$ does not contain any points but it does contain many points if you include all prime ideals.*

Example 7.12. *Now suppose that we take $\mathbb{Z}$. In this case the maximal ideals correspond to the prime numbers, and in addition there is one point whose closure is the whole spectrum. In this respect $\operatorname{Spec} \mathbb{Z}$ is very similar to $\operatorname{Spec} k[t]$.*

We will need one very useful fact from commutative algebra:

Lemma 7.13. *If $\mathfrak{a} \trianglelefteq R$ is an ideal in a ring R then the radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$.*

Proof. One inclusion is clear; every prime ideal $\mathfrak{p}$ is radical (that is, equal to its own radical) and so the intersection of all prime ideals containing $\mathfrak{a}$ is radical.

Now suppose that r does not belong to the radical of $\mathfrak{a}$. Let $\mathfrak{b}$ be the ideal generated by the image of $\mathfrak{a}$ inside the ring R_r. Then the image of r inside the quotient ring $R_r/\mathfrak{b}$ is non-zero. Pick an ideal in this ring, maximal with respect to the property that it does not contain the image of r. Then the inverse image $\mathfrak{p}$ of this ideal is a prime ideal which does not contain r. □

Lemma 7.14. *Let X be the spectrum of the ring R and let $f \in R$.*

If $U_f = \bigcup U_{g_i}$ then $f^n = \sum b_i g_i$, where $b_1, b_2, \dots, b_k \in R$. In particular, U_f is a finite union of the U_{g_i} and U_f is compact.

Proof. Taking complements, we see that

$$V(\langle f \rangle) = \bigcap_i V(\langle g_i \rangle) = V(\langle \sum_i g_i \rangle).$$

Now $V(\mathfrak{a})$ consists of all prime ideals that contain $\mathfrak{a}$, and the radical of $\mathfrak{a}$ is the intersection of all the prime ideals that contain $\mathfrak{a}$. Thus

$$\sqrt{\langle f \rangle} = \sqrt{\langle \sum_i g_i \rangle}.$$

But then, in particular, f^n is a finite linear combination of the g_i and the corresponding open sets cover U_f. □