4. Morphisms

We adopt the following working definition of a morphism between affine varieties.

Definition 4.1. *A* ***morphism***

$$f\colon V \longrightarrow W,$$

between two affine varieties, where $V \subset \mathbb{A}^m$ and $W \subset \mathbb{A}^n$ is given by picking a collection of n polynomials $f_1, f_2, \dots, f_n \in K[x_1, x_2, \dots, x_m]$ such that

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x)) \in W,$$

for every $x \in V$.

Note that this gives us a category. The category of affine varieties, with maps given by morphisms.

Example 4.2. *The map*

$$f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2,$$

given by

$$t \longrightarrow (t, t^2)$$

is a morphism.

It is interesting and instructive to look at the image. If we pick coordinates (x, y) on $\mathbb{A}^2$ then it is not too hard to check that the image is $C = V(y - x^2)$.

Now consider the morphism

$$C \longrightarrow \mathbb{A}^1 \qquad \text{given by} \qquad (x, y) \longrightarrow x.$$

This is a morphism and in fact it is pretty easy to see that the composition either way is the identity, so that it is the inverse of the first morphism and C is isomorphic to $\mathbb{A}^1$.

This example has many interesting generalisations. For example we can look at the morphism:

Example 4.3.

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^3,$$

given as

$$t \longrightarrow (t, t^2, t^3)$$

The image is $C = V(y - x^2, z - x^3)$ and projection down to the x-axis is the inverse map.

It turns out that both the nodal and cuspidal cubic can be parametrised.

Example 4.4. *Consider the map*

$$f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2,$$

given by

$$t \longrightarrow (t^2-1, t(t^2-1))$$

The image lies in the nodal cubic:

$$C = V(y^2 - x^2 - x^3) \subset \mathbb{A}^2,$$

since

$$x^2 + x^3 = x^2(x+1) = (t^2-1)^2 t^2 = t(t^2-1)^2 = y^2.$$

But in fact, if $y^2 = x^2 + x^3$ then either $x = 0$ in which case $y = 0$ and this is the image of $t = \pm 1$ or we put $t = y/x$, in which case it is easy to check that $f(t) = (x, y)$.

Example 4.5. *Consider the map*

$$g\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2,$$

given by

$$t \longrightarrow (t^2, t^3)$$

The image lies in the cuspidal cubic:

$$C = V(y^2 - x^3) \subset \mathbb{A}^2.$$

But in fact, if $y^2 = x^3$ then either $x = 0$ in which case $y = 0$ and this is the image of $t = 0$ or we put $t = y/x$, in which case it is easy to check that $f(t) = (x, y)$.

In all of these examples, note that the image is a closed subset. However

Example 4.6. *Let $C = V(xy-1) \subset \mathbb{A}^2$ and consider the morphism*

$$C \longrightarrow \mathbb{A}^1 \qquad \text{given by} \qquad (x, y) \longrightarrow x.$$

The image is the set of all non-zero elements of $\mathbb{A}^1$. This is an open subset not a closed subset.

Definition 4.7. *If $U \subset V \subset \mathbb{A}^n$ is an open subset of the affine subvariety V then we say that U is **quasi-affine**.*

Remark 4.8. *Note that an open subset of a closed subset is the same as a locally closed subset (that is, U is the intersection of an open subset and a closed subset).*

It seems natural to wonder if quasi-affine varieties are closed under taking the image.

Example 4.9. *Consider the morphism*

$$f\colon \mathbb{A}^2 \longrightarrow \mathbb{A}^2 \qquad \text{given by} \qquad (x, y) \longrightarrow (xy, y).$$

Let $(a, b) \in \mathbb{A}^2$ be a point in the target $\mathbb{A}^2$. If $b \neq 0$ then $(x, y) = (a/b, b)$ maps to (a, b). If $f(x, y) = (a, 0)$ then $y = 0$ so that $a = xy = 0$. Thus the image consists of the open subset where $y \neq 0$ union the origin. It is not hard to see that this is not quasi-affine, that is this set is not the intersection of an open and a closed set.

How should one define morphisms for quasi-affine varieties? Well, the natural thing to do is to restrict morphisms from the closure, which is an affine variety. But consider the hyperbola, $C = V(xy-1)$. The map $(x, y) \longrightarrow x$ is equivalent to the map $(x, y) \longrightarrow 1/y$, since $x = 1/y$ on C. So it is more natural to define a morphism as something which is locally the restriction of a collection of rational functions.

Definition 4.10. *Let*

$$f\colon X \longrightarrow Y,$$

be a map between two quasi-affine varieties $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$. We say that f is a morphism, if there is an open affine cover U_i for X such that for every i the restriction $f_i = f|_{U_i}$

$$f_i\colon U_i \longrightarrow Y,$$

is given by a collection of rational functions, g_j/h_j,

$$(x_1, x_2, \dots, x_m) \longrightarrow (g_1/h_1, g_2/h_2, \dots g_n/h_n)$$

where none of the h_j are zero on U_i.

Definition-Lemma 4.11. *Let $V \subset \mathbb{A}^n$ be an affine variety and let f be a polynomial. Let*

$$U_f = \{\, x \in V \mid f(x) \neq 0 \,\}.$$

Then the quasi-affine variety U_f is isomorphic to an affine variety.

Proof. Let $W \subset \mathbb{A}^{n+1}$ be the closed subset defined by the ideal of V together with the polynomial $yf - 1$. Define a morphism

$$\phi\colon W \longrightarrow V,$$

given by

$$(x_1, x_2, \dots, x_n, y) \longrightarrow (x_1, x_2, \dots, x_n).$$

The image is contained in U_f, since if we can find y such that $yf(x) = 1$ then $f(x) \neq 0$.

Define a morphism

$$U_f \longrightarrow W,$$

given by

$$\psi\colon (x_1, x_2, \dots, x_n) \longrightarrow (x_1, x_2, \dots, x_n, 1/f).$$

Then it is not hard to see that ϕ and ψ are inverses of each other. $\square$

Definition-Lemma 4.12. *Let X be a quasi-affine variety. A **regular function** is a morphism $X \longrightarrow \mathbb{A}^1 = K$.*

The set of all regular functions $\mathcal{O}_X(X)$ is a K-algebra.

Proof. Clear. $\square$

It seems worthwhile to point out that at this point one can state the very famous:

Conjecture 4.13 (Jacobian conjecture)**.** *Let*

$$\phi\colon \mathbb{A}^2 \longrightarrow \mathbb{A}^2,$$

be a morphism, given by

$$(x, y) \longrightarrow (f, g).$$

ϕ is an isomorphism if and only if the Jacobian

$$J(\phi) = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$$

is a non-zero constant.

Remark 4.14. *It is easy to show that if ϕ is an isomorphism then the Jacobian $J(\phi)$ is a non-zero constant.*