3. Categories and Functors

We recall the definition of a category:

Definition 3.1. *A category $\mathcal{C}$ is the data of two collections. The first collection is called the* ***objects*** *of $\mathcal{C}$ and is denoted* $\mathrm{Obj}(\mathcal{C})$*. Given two objects X and Y of $\mathcal{C}$, we associate another collection* $\mathrm{Hom}(X, Y)$*, called the* ***morphisms*** *between X and Y. Further we are given a law of* ***composition*** *for morphisms: given three objects X, Y and Z, there is an assignment*

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z).$$

Given two morphisms, $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, Z)$, $g \circ f \in \mathrm{Hom}(X, Z)$ denotes the composition. Further this data satisfies the following axioms:

(1) *Composition is associative,*

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

for all objects X, Y, Z, W and all morphisms $f \colon X \longrightarrow Y$, $g \colon Y \longrightarrow Z$ and $h \colon Z \longrightarrow W$.

(2) *For every object X, there is a special morphism $i = i_X \in \mathrm{Hom}(X, X)$ which acts as an identity under composition. That is for all $f \in \mathrm{Hom}(X, Y)$,*

$$f \circ i_X = f = i_Y \circ f.$$

We say that a category $\mathcal{C}$ is called ***locally small*** *if the collection of morphisms is a set. If in addition the collection of objects is a set, we say that $\mathcal{C}$ is* ***small****.*

There are an abundance of categories:

Example 3.2. *The category (Sets) of sets and functions; the category of (Groups) groups and group homomorphisms; the category (Vec) of vector spaces and linear maps; the category (Top) of topological spaces and continuous maps; the category (Rings) of rings and ring homomorphisms. All of these are locally small categories.*

Let X be a topological space. We can define a small category $\mathfrak{Top}X$ associated to X as follows. The objects of $\mathfrak{Top}X$ are simply the open subsets of X. Given two open subsets U and V,

$$\mathrm{Hom}(U, V) = \begin{cases} i_{UV} & \text{if } U \subset V \\ \emptyset & \text{otherwise.} \end{cases}$$

Here i_{UV} is a formal symbol. Composition of morphisms is defined in the obvious way (in fact the definition is forced, there are no choices to be made).

Definition 3.3. *We say that a category $\mathcal{D}$ is a subcategory of $\mathcal{C}$ if every object of $\mathcal{D}$ is an object of $\mathcal{C}$ and for every pair of objects X and Y of $\mathcal{D}$, $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ is a subset of $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ (that is, every morphism in $\mathcal{D}$ is a morphism in $\mathcal{C}$). The identity and composition of morphisms should come out the same.*

We say that $\mathcal{D}$ is a ***full subcategory*** *of $\mathcal{C}$, if for every pair of objects X and Y of $\mathcal{D}$, $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ is equal to $\operatorname{Hom}_{\mathcal{C}}(X, Y)$.*

The category of finite sets is a full subcategory of the category ($\underline{\text{Sets}}$) of sets. Similarly the category of finite dimensional linear spaces is a full subcategory of the category ($\underline{\text{Vec}}$) of vector spaces. By comparison the category ($\underline{\text{Groups}}$) of groups is a subcategory of the category ($\underline{\text{Sets}}$) of sets (this example is a bit of a cheat) but it is not a full subcategory. In other words not every function is a group homomorphism.

It is easy construct new categories from old ones:

Definition 3.4. *Given a category $\mathcal{C}$, the* ***opposite category****, denoted $\mathcal{C}^{\text{op}}$, is the category, whose objects are the same as $\mathcal{C}$, but whose morphisms go the other way, so that*

$$\operatorname{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X).$$

Definition 3.5. *The* ***inverse*** *of a morphism $f\colon X \longrightarrow Y$ is a morphism $g\colon Y \longrightarrow X$, such that $f \circ g$ and $g \circ f$ are both the identity map. If the inverse of f exists, then we say that f is an* ***isomorphism*** *and that X and Y are* ***isomorphic****.*

Definition 3.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A* ***covariant functor*** *from F from $\mathcal{C}$ to $\mathcal{D}$ assigns to every object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$ and to every morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \longrightarrow F(Y)$ in $\mathcal{C}$, compatible with composition and the identity.*

That is

$$F(g \circ f) = F(g) \circ F(f) \qquad \text{and} \qquad F(i_X) = i_{F(X)}.$$

A ***contravariant functor*** *F is the same as covariant functor, except that arrows are reversed,*

$$F(f)\colon F(Y) \longrightarrow F(X),$$

and

$$F(g \circ f) = F(f) \circ F(g).$$

In other words a contravariant functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the same as a covariant functor $F\colon \mathcal{C}^{\text{op}} \longrightarrow \mathcal{D}$

It is easy to give examples of functors. Let

$$F\colon (\underline{\text{Rings}}) \longrightarrow (\underline{\text{Groups}}),$$

be the functor which assigns to every ring R, the underlying additive group, and to every ring homomorphism f, the corresponding group homomorphism (the same map of course).

It is easy to check that F is indeed a functor; for obvious reasons it is called a forgetful functor and there are many such functors.

Note that we may compose functors in the obvious way and that there is an identity functor. Slightly more interestingly there is an obvious contravariant functor from a category to its opposite.

There are three non-trivial well-known functors. First there is a functor, denoted H_*, from the category $(\underline{\text{Top}})$ of topological spaces to the category of (graded) groups, which assigns to every topological space its singular homology. Similarly there is a contravariant functor from category $(\underline{\text{Top}})$ of topological spaces to the category of (graded) rings, which assigns to every topological space its singular cohomology.

The second and third are much more general.

Definition 3.7. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We say that F is **faithful** if for every f and g, morphisms in $\mathcal{C}$, $F(f) = F(g)$ iff $f = g$. We say that F is **full** if for every morphism $h\colon F(X) \longrightarrow F(Y)$ in $\mathcal{D}$, there is a morphism f in $\mathcal{C}$ such that $F(f) = h$. We say that F is **essentially surjective** if for every object A in $\mathcal{D}$ there is an object X in $\mathcal{C}$ such that A is isomorphic to $F(X)$.*

*We say that F is an **equivalence of categories** if F is fully faithful and essentially surjective.*

For example, let $\mathcal{D}$ be the category of finite dimensional vector spaces over a field K. Let $\mathcal{C}$ be the category whose objects are the natural numbers, and such that the set of morphisms between two natural numbers m and n, is equal to the set of $m \times n$ matrices, with the obvious rule of composition. Then $\mathcal{C}$ is naturally a full subcategory of $\mathcal{D}$ (assign to n the "standard" vector space K^n) and the inclusion map, considered as a functor, is an equivalence of categories. Note however that there is no functor the other way.

More generally, given a category $\mathcal{D}$, one may form a quotient category $\mathcal{C}$. Informally the objects and morphisms of $\mathcal{C}$ are equivalence classes of objects of $\mathcal{D}$, under isomorphism.

We now turn to the third important functor. We first note that given two categories $\mathcal{C}$ and $\mathcal{D}$, where $\mathcal{C}$ is locally small, the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ is a category, denoted $\text{Fun}(\mathcal{C}, \mathcal{D})$. The objects of this category are simply functors from $\mathcal{C}$ to $\mathcal{D}$. Given two functors F and G, a morphism between them is a natural transformation:

Definition 3.8. *Let F and G be two functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. A **natural transformation** u from F to G assigns to* *every object X of $\mathcal{C}$ a morphism $u_X\colon F(X) \longrightarrow G(X)$ such that for every morphism $f\colon X \longrightarrow Y$ in $\mathcal{C}$ the following diagram commutes*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle u_X}\big\downarrow & & {\scriptstyle u_Y}\big\downarrow \\ G(X) & \xrightarrow{G(f)} & G(Y). \end{array}$$

It is easy to check that we may compose natural transformations, that this composition is associative and that the natural transformation which assigns to every object X, the identity map from $F(X)$ to $F(X)$ acts as an identity, so that $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is indeed a category.

Suppose that $\mathcal{C}$ is a locally small category. Let Y be an object of $\mathcal{C}$. I claim that we get a functor $h_Y\colon \mathcal{C} \longrightarrow (\underline{\text{Sets}})$. Given an object X of $\mathcal{C}$, we associate the set $h_Y(X) = \operatorname{Hom}(X, Y)$. Given a morphism $f\colon X \longrightarrow X'$, note that we get a map

$$h_Y(f) : \operatorname{Hom}(X', Y) \longrightarrow \operatorname{Hom}(X, Y),$$

which takes a morphism g and assigns the morphism $h_Y(f)(g) = g \circ f$. It is easy to check that h_Y is a contravariant functor. On the other hand, varying Y, I claim we get a functor

$$h\colon \mathcal{C} \longrightarrow \operatorname{Fun}(\mathcal{C}^{\text{op}}, (\underline{\text{Sets}})).$$

At the level of objects, the definition of this functor is obvious. Given $Y \in \mathcal{C}$ we assign the object $h_Y \in \operatorname{Fun}(\mathcal{C}^{\text{op}}, (\underline{\text{Sets}}))$. On the other hand, given a morphism $f\colon Y \longrightarrow Y'$, I claim that we get a natural transformation $h(f)$ between the two functors h_Y and $h_{Y'}$ going from $\mathcal{C}^{\text{op}}$ to $(\underline{\text{Sets}})$. Thus given an object X in $\mathcal{C}$, we are supposed to give a morphism

$$h(f)_X\colon h_Y(X) = \operatorname{Hom}(X, Y) \longrightarrow h_{Y'}(X) = \operatorname{Hom}(X, Y').$$

The definition of $h(f)_X$ is clear. Given $g \in \operatorname{Hom}(X, Y)$, send this to $h(f)_X(g) = f \circ g$. It is easy to check that $h(f)$ is indeed a natural transformation and that h is a functor. More significantly:

Theorem 3.9 (Yoneda's Lemma)**.** *h is fully faithful.*

The proof is left as an exercise for the reader. Yoneda's Lemma thus says that if we want to understand the category $\mathcal{C}$, we can think of it as a subcategory of the category of contravariant functors from $\mathcal{C}$ to the category $(\underline{\text{Sets}})$ of sets.

In these terms obviously the must fundamental question is to ask which of these functors is in the image:

Definition 3.10. *We say that the functor $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow (\underline{Sets})$ is* ***representable*** *(by Y) if it is isomorphic to h_Y, for some object Y of $\mathcal{C}$.*

By Yoneda's Lemma, if F is representable by Y then Y is determined up to unique isomorphism.