18. Fibre products of schemes

The main result of this section is:

Theorem 18.1. *The category of schemes admits fibre products.*

A key part of the proof is to pass from the local case (in which case all three schemes are affine) to the global case. To do this, we need to be able to construct morphisms, by constructing them locally. We will need:

Theorem 18.2. *Let $f_i\colon U_i \longrightarrow Y$ be a collection of morphisms of schemes, with a varying domain, but a fixed target.*

Suppose that for each pair of indices i and j, we are given open subsets $U_{ij} \subset U_i$, and isomorphisms $\phi_{ij}\colon U_{ij} \longrightarrow U_{ji}$, such that $f_i|_{U_{ij}} = f_j \circ \phi_{ij}\colon U_{ij} \longrightarrow Y$ and

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $U_{ij} \cap U_{ik}$, for all i, j and k (we adopt the convention that $U_{ii} = U_i$, so that ϕ_{ii} is the identity and moreover $\phi_{ij}^{-1} = \phi_{ji}$).

Then there is a morphism of schemes $f\colon X \longrightarrow Y$, open immersions $\psi_i\colon U_i \longrightarrow X$, whose images cover X, such that $f_i = f \circ \psi_i\colon U_i \longrightarrow Y$ and $\psi_i|_{U_{ij}} = \psi_j \circ \phi_{ij}\colon U_{ij} \longrightarrow Y$.

X is unique, up to unique isomorphism, with these properties.

We prove (18.2) in two steps (one of which can be further broken down into two substeps):

- Construct the scheme X.
- Construct the morphism f.

In fact, having constructed X, it is straightforward to construct f. Since a scheme consists of two parts, a topological space and a sheaf, we can break the first step into two smaller pieces:

- Construct the underlying topological space.
- Construct the structure sheaf.

We first show how to patch a sheaf, which is the hardest part:

Lemma 18.3. *Let X be a topological space, and let $\{X_i\}$ be an open cover of X. Suppose that we are given sheaves $\mathcal{F}_i$ on X_i and for each i and j an isomorphism*

$$\phi_{i,j}\colon \mathcal{F}_i|_{X_{ij}} \longrightarrow \mathcal{F}_j|_{X_{ij}},$$

such that

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the triple intersection X_{ijk}, for all i, j and k.

Then there is a sheaf $\mathcal{F}$ on X, together with isomorphisms,

$$\psi_i\colon \mathcal{F}|_{X_i} \longrightarrow \mathcal{F}_i,$$

which satisfy $\psi_j = \phi_{ij} \circ \psi_i$. Further $\mathcal{F}$ is unique up to unique isomorphism, with these properties.

Proof. We just show how to define $\mathcal{F}$ and leave the rest to the interested reader. Let $U \subset X$ be any open set, and let $U_i = U \cap X_i$.

$$\mathcal{F}(U) = \{\, (s_i) \in \prod_i \mathcal{F}_i(U_i) \,|\, \phi_{ij}(s_i|_{U_{ij}}) = s_j|_{U_{ji}} \,\}. \qquad \square$$

Using (18.3), one can put a natural scheme structure on any closed subset of a scheme (natural means the smallest possible scheme structure):

Definition-Lemma 18.4. *Let X be scheme and let Y be a closed subset. Then Y has a unique reduced subscheme structure, called the* ***reduced induced subscheme structure****.*

Proof. We first assume that $X = \operatorname{Spec} A$ is affine. Let $\mathfrak{a}$ be the ideal obtained by intersecting all the prime ideals in Y. Then $\mathfrak{a}$ is the largest ideal for which $V(\mathfrak{a}) = Y$. The induced scheme structure on Y is reduced, that is, the stalks of $\mathcal{O}_Y$ have no nilpotent elements, as $\mathfrak{a}$ is a radical ideal.

Now suppose that X is an arbitrary scheme. For each open affine subset $U_i \subset X$, let $Y_i \subset U_i$ be the reduced induced subscheme structure on $Y \cap U_i$. This gives us a sheaf $\mathcal{O}_{Y_i}$ on each Y_i and we want to construct a sheaf $\mathcal{O}_Y$ on the whole of Y. By (18.3) it suffices to prove that the sheaves $\mathcal{O}_{Y_i}$ agree on overlaps.

It is not hard to reduce to the case where $U = \operatorname{Spec} A$, $V = \operatorname{Spec} A_f$. We want to show that the reduced induced subscheme structure on V is the same as restricting the reduced induced subscheme structure from U to V. But this is the same as to say that if $\mathfrak{a}$ is the intersection of those prime ideals of A which are contained in Y, then $\mathfrak{a}A_f$ is the intersection of those prime ideals of A_f which are contained in Y, which is clear. $\square$

The next step is to bump this up to schemes:

Lemma 18.5. *Suppose that we are given schemes U_i, and subschemes $U_{ij} \subset U_i$, together with isomorphisms,*

$$\phi_{ij}\colon U_{ij} \longrightarrow U_{ji},$$

which satisfy

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $U_{ij} \cap U_{ik}$*, for all* i*,* j *and* k*.*

Then there is a scheme X *and open immersions* $\psi_i \colon U_i \longrightarrow X$*, whose images cover* X*, which satisfy* $\psi_i|_{U_{ij}} = \psi_i \circ \phi_{ij} \colon U_{ij} \longrightarrow X$*.*

Proof. We first construct the topological space X. Let

$$X = \coprod_i U_i / \sim \qquad \text{where} \qquad x_i \in U_{ij} \sim x_j \in U_{ji} \text{ iff } \phi_{ij}(x_i) = x_j.$$

Here $\sim$ denotes the equivalence relation generated by the rule on the RHS, and X is just the quotient topological space (which always exists). Note that

$$X_i = U_i / \sim,$$

is an open subset of X and there are homeomorphisms $\phi_i \colon U_i \longrightarrow X_i$. Now construct a sheaf $\mathcal{O}_X$ on X, using (18.3). This gives us a locally ringed space $(X, \mathcal{O}_X)$ and the remaining properties can be easily checked. □

There are a couple of interesting examples of the construction of schemes. The first is to take U_{ij} empty (so that there are no patching conditions at all). The resulting scheme is called the **disjoint union** and is denoted

$$\coprod_i X_i.$$

Another more interesting example proceeds as follows. Take two copies U_1 and U_2 of the affine line. Let $U_{12} = U_{21}$ be the complement of the origin, and let ϕ_{12} be the identity. Then X is obtained by identifying every point, except the origin. Note that this is like the classical construction of a topological space, which is locally a manifold, but which is not Hausdorff. Of course no scheme is ever Hausdorff (apart from the most trivial examples) and it turns out that there is an appropriate condition for schemes (and in fact morphisms of schemes) which is a replacement for the Hausdorff condition for topological spaces.

Finally we turn to the problem of glueing morphisms, which is the easiest bit:

Proof of (18.2). Let X be the scheme constructed in (18.5). It is clear that to give a morphism $f \colon X \longrightarrow Y$ is the same as to give morphisms $f_i \colon X_i \longrightarrow Y$, compatible on overlaps. □

Lemma 18.6. *Let* X *and* Y *be schemes over* S*. Suppose that* X *has an open cover* $\{X_i\}$ *such that the fibre product* $F_i = X_i \underset{S}{\times} Y$ *exists.*

Then the fibre product $F = X \underset{S}{\times} Y$ *exists.*

Proof. Let $p_i\colon F_i \longrightarrow X$ be the natural morphism and let $F_{ij} = p_i^{-1}(X_j)$. Note that F_{ij} is isomorphic to the fibre product of $X_i \cap X_j$ and Y over S. Indeed if Z maps to X_{ij} and Y over S, it maps to X_i and Y over S. But then Z maps to F_i, by the universal property of the fibre product. It is clear that the image of Z lands in F_{ij}, so that F_{ij} is the fibre product. But then there are natural isomorphisms $\phi_{ij}\colon F_{ij} \longrightarrow F_{ji}$ such that

$$\phi_{ik} = \phi_{jk} \circ \phi_{ij},$$

on the intersection $F_{ij} \cap F_{ik}$, for all i, j and k, and $p_i|_{F_{ij}} = p_j \circ \phi_{ij}$.

(18.2) implies that we may patch F_i to a scheme F, and patch the morphisms $F_i \longrightarrow X$ to a morphism $F \longrightarrow X$. Similarly we may construct a morphism $F \longrightarrow Y$, from the individual morphisms $q_i\colon F_i \longrightarrow Y$.

Now suppose we are given $Z \longrightarrow X$ and $Z \longrightarrow Y$ morphisms over S. The open cover $\{X_i\}$ induces an open cover $\{Z_i\}$ of Z. We get morphisms $Z_i \longrightarrow F_i$, by the universal property of F_i and so we get morphisms $Z_i \longrightarrow F$ by composition. It is easy to check that these patch to a morphism $Z \longrightarrow F$. But then F is the fibre product. □

Proof of (18.1). Let X and Y be two schemes over S. We want to construct the fibre product.

First suppose that $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $S = \operatorname{Spec} R$. Then there are ring homomorphisms $R \longrightarrow A$ and $R \longrightarrow B$ and so A and B are R-algebras. As $C = A \underset{R}{\otimes} B$ is the pushout in the category of rings, it follows that $Z = \operatorname{Spec} C$ is the fibre product in the category of affine schemes; in fact it is also the fibre product in the category of schemes, since a morphism to an affine scheme is the same as a ring homomorphism the other way on global sections.

We now bump this result up to the global case. First suppose that S and Y are affine. Since an arbitrary scheme X can be covered by open affines $\{X_i\}$, (18.6) implies that the fibre product of X and Y over S exists.

Now suppose that S is affine. Since Y can be covered by open affines $\{Y_j\}$ and the fibre product is obviously symmetric in X and Y, (18.6) implies that the fibre product of X and Y over S exists.

Now take an affine cover S_i of S. Let X_i and Y_i be the inverse image of S_i (meaning take the open subscheme on the open set $p_j^{-1}(S_i)$). Then the fibre product $X_i \underset{S}{\times} Y_i$ exists. But in fact this is also a fibre product for $X_i \underset{S}{\times} Y$, since anything lying over X_i automatically lies over Y_i. Since X_i forms an open cover of X we are done by one more application of (18.6). □