16. More examples of schemes

Definition 16.1. *Let X be a scheme and let $x \in X$ be a point of X. The **residue field of** X **at** x is the quotient of $\mathcal{O}_{X,x}$ by its maximal ideal.*

We recall some basic facts about valuations and valuation rings.

Definition 16.2. *Let K be a field and let G be a totally ordered abelian group. A **valuation** of K with values in G, is a map*

$$\nu \colon K - \{0\} \longrightarrow G,$$

such that for all x and $y \in K - \{0\}$ we have:

(1) $\nu(xy) = \nu(x) + \nu(y)$.
(2) $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

Definition-Lemma 16.3. *If ν is a valuation, then the set*

$$R = \{ x \in K \mid \nu(x) \geq 0 \} \cup \{0\},$$

*is a subring of K, which is called the **valuation ring** of ν. The set*

$$\mathfrak{m} = \{ x \in K \mid \nu(x) > 0 \} \cup \{0\},$$

is an ideal in R and the pair $(R, \mathfrak{m})$ is a local ring.

Proof. Easy check. □

Definition 16.4. *A valuation is called a **discrete valuation** if $G = \mathbb{Z}$ and ν is surjective. The corresponding valuation ring is called a **discrete valuation ring**. Any element $t \in R$ such that $\nu(t) = 1$ is called a **uniformising parameter**.*

Lemma 16.5. *Let R be an integral domain, which is not a field.*

The following are equivalent:

- *R is a DVR.*
- *R is a local ring and a PID.*

Proof. Suppose that R is a DVR. Then R is certainly a local ring. Suppose that a and $b \in R$ and $\nu(a) = \nu(b)$. Then $\nu(b/a) = \nu(b) - \nu(a) = 0$ and so $\langle a \rangle = \langle b \rangle$. It follows that the ideals of R are of the form

$$I_k = \{ a \in R \mid \nu(a) \geq k \}.$$

As ν is surjective, there is an element $t \in R$ such that $\nu(t) = 1$. Then

$$I_k = \langle t^k \rangle = \mathfrak{m}^k.$$

Thus R is a PID.

Now suppose that R is a local ring and a PID. Let $\mathfrak{m}$ be the unique maximal ideal. As R is a PID, $\mathfrak{m} = \langle t \rangle$, for some $t \in R$. Define a map

$$\nu \colon K \longrightarrow \mathbb{Z},$$

by sending a to k, where $a \in \mathfrak{m}^k - \mathfrak{m}^{k+1}$ and extending this to any fraction a/b in the obvious way. It is easy to check that ν is a valuation and that R is the valuation ring. □

There are two key examples of a DVR. First let k be field and let $R = k[t]_{\langle t \rangle}$. Then R is a local ring and a PID so that R is a DVR. t is a uniformising parameter. Note that R is the stalk of the struture sheaf of the affine line at the origin.

Now let

$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \},$$

be the unit disc in the complex plane. Then the stalk $\mathcal{O}_{\Delta,0}$ of the sheaf of holomorphic functions is a local ring. The order of vanishing realises this ring as a DVR. z is a uniformising parameter.

In fact if C is a smooth algebraic curve, an algebraic variety of dimension one, then $\mathcal{O}_{C,p}$ is a DVR.

Example 16.6. *Let R be the local ring of a curve over an algebraically closed field (or more generally a discrete valuation ring). Then* $\operatorname{Spec} R$ *consists of two points; the maximal ideal, and the zero ideal. The first t_0 is closed and has residue field the groundfield k of C, the second t_0 has residue field the quotient ring K of R, and its closure is the whole of X. The inclusion map $R \longrightarrow K$ corresponds to a morphism which sends the unique point of* $\operatorname{Spec} K$ *to t_1.*

Example 16.7. *There is another morphism of ringed spaces which sends the unique point of* $\operatorname{Spec} K$ *to t_0 and uses the inclusion above to define the map on structure sheaves.*

Since there is only one way to map R to K, this does not come from a map on rings. In fact the second map is not a morphism of locally ringed spaces, and so it is not a morphism of schemes.

It is interesting to see an example of an affine scheme, in a seemingly esoteric case. Consider the case of a number field k (that is, a finite extension of $\mathbb{Q}$, with its ring of integers $A \subset k$, that is, the integral closure of $\mathbb{Z}$ inside k). As a particular example, take

Example 16.8. $k = \mathbb{Q}(\sqrt{3})$. *Then $A = \mathbb{Z} \oplus \mathbb{Z}\langle\sqrt{3}\rangle$. The picture is very similar to the case of $\mathbb{Z}$. There are infinitely many maximal ideals, and only one point which is not closed, the zero ideal. Moreover, as there is a natural ring homomorphism $\mathbb{Z} \longrightarrow A$, by our equivalence* *of categories, there is an induced morphism of schemes* $\operatorname{Spec} A \longrightarrow \operatorname{Spec} \mathbb{Z}$. *We investigate this map. Consider the fibre over a point* $\langle p \rangle \in \operatorname{Spec} \mathbb{Z}$. *This is just the set of primes in* A *containing the ideal* pA. *It is well known by number theorists, that three things can happen:*

(1) If p *divides the discriminant of* $k/\mathbb{Q}$ *(which in this case is* 12*), that is,* $p = 2$ *or* 3, *then the ideal* $\langle p \rangle$ *is a square in* A.

$$\langle 2 \rangle A = \langle -1 + \sqrt{3} \rangle^2,$$

and

$$\langle 3 \rangle A = \langle \sqrt{3} \rangle^2.$$

(2) If 3 *is a square modulo* p, *the prime* $\langle p \rangle$ *factors into a product of distinct primes,*

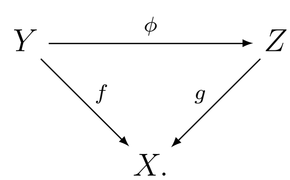
$$\langle 11 \rangle A = \langle 4 + 3\sqrt{3} \rangle \langle 4 - 3\sqrt{3} \rangle,$$

or

$$\langle 13 \rangle A = \langle 4 + \sqrt{3} \rangle \langle 4 - \sqrt{3} \rangle,$$

(3) If $p > 3$ *and* 3 *is not a square mod* p *(e.g* $p = 5$ *and* 7*), the ideal* $\langle p \rangle$ *is prime in* A.

Definition 16.9. *Let* $\mathcal{C}$ *be a category and let* X *be an object of* $\mathcal{C}$. *Let* $\mathcal{D} = \mathcal{C}|_X$ *be the category whose objects consist of pairs* $f \colon Y \longrightarrow X$, *where* f *is a morphism of* $\mathcal{C}$, *and whose morphisms, consist of commutative diagrams*



$\mathcal{D}$ *is known as the category over* X. *If* X *is a scheme, then a scheme over* X *is exactly an object of the category of schemes over* X. *Let* R *be a ring.* ***Affine*** n***-space over*** R, *denoted* $\mathbb{A}^n_R$, *is the spectrum of the polynomial ring* $R[x_1, x_2, \dots, x_n]$.

One of the key ideas of schemes, is to work over arbitrary bases. Note that since there is an inclusion $R \longrightarrow R[x_1, x_2, \dots, x_n]$ of rings, affine space over R is a scheme over $\operatorname{Spec} R$. Thus we may define affine space over any affine scheme.