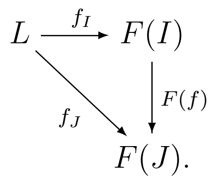
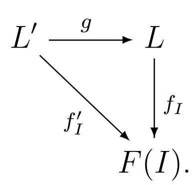
14. Limits

One of the more interesting notions of category theory, is the theory of limits.

Definition 14.1. *Let $\mathbb{I}$ be a category and let $F\colon \mathbb{I} \longrightarrow \mathcal{C}$ be a functor. A* ***prelimit*** *for F is an object L of $\mathcal{C}$, together with morphisms $f_I\colon L \longrightarrow F(I)$, for every object I of $\mathbb{I}$, which are compatible in the following sense: Given a morphism $f\colon I \longrightarrow J$ in $\mathbb{I}$, the following diagram commutes*



The ***limit of*** *F, denoted $L = \varprojlim_{\mathbb{I}} F$ is a prelimit L, which is universal amongst all prelimits in the following sense: Given any prelimit L' there is a unique morphism $g\colon L' \longrightarrow L$, such that for every object I in $\mathbb{I}$, the following diagram commutes*



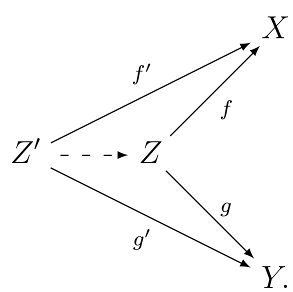
Informally, then, if we think of a prelimit as being to the left of every object $F(I)$, then the limit is the furthest prelimit to the right. Note that limits, if they exist at all, are unique, up to unique isomorphism, by the standard argument. Note also that there is a dual notion, the notion of colimits. In this case, F is a contravariant functor and all the arrows go the other way (informally, then, a prelimit is to the right of every object $F(I)$ and a limit is any prelimit which is furthest to the left).

Let us look at some special cases. First suppose we take for $\mathbb{I}$ the category with one object and one morphism. In this case a functor picks out an object. It is clear that in this case the limit is the same object. Similarly for the colimit.

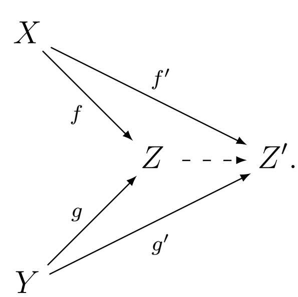
It is in fact more interesting to take for $\mathbb{I}$ the empty category, that is the category with no objects and no morphisms. Then every object is a prelimit and so a limit has the property that every object has a unique map to it. For obvious reasons this is called a **terminal object**. The category (Sets) of sets has as terminal object any set with one object; the category (Vec) of vector spaces any space of dimension zero. The colimit has the property that it has a unique map to every object and it is called an **initial object**. The empty set is an initial object of the category (Sets) of sets; the group with one element is an initial object in the category (Groups) of groups.

At the other extreme one can take the identity functor, so that $\mathbb{I} = \mathcal{C}$. A limit, if it exists at all, is an object to which all other objects map (in a compatible fashion). In the case that a category has an initial object, then the limit of the identity functor is the initial object. Dually, a colimit, if it exists at all, is an object which maps to all other objects. In the case that a category has a terminal object, then the colimit of the identity functor is the terminal object.

Now take as category two objects, with two morphisms (that is, the two identity maps). A functor picks out two objects, call them X and Y. First consider the case of the limit. A prelimit is the data of an object Z, together with a pair or morphisms, $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$. This prelimit is a limit if and only if it is universal amongst all such prelimits. That is, suppose we are given two morphisms $f'\colon Z' \longrightarrow X$ and $g'\colon Z' \longrightarrow Y$, then there is a unique induced morphism $h\colon Z' \longrightarrow Z$, such that the following diagram commutes



Dually, consider the case of a colimit, where all the arrows are reversed. A prelimit is the data of an object Z, together with a pair of morphisms, $f\colon X \longrightarrow Z$ and $g\colon Y \longrightarrow Z$. This prelimit is a limit if and only if it is universal amongst all such prelimits. That is, suppose we are given two morphisms $f'\colon X \longrightarrow Z'$ and $g'\colon Y \longrightarrow Z'$, then there is a unique induced morphism $h\colon Z \longrightarrow Z'$, such that the following diagram commutes

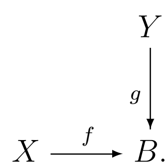


Definition 14.2. *Let X and Y be two objects of a category $\mathcal{C}$. The* ***product*** *is the limit and the* ***coproduct*** *is the colimit, of the functor above.*

The product of two sets is the ordinary cartesian product; the product of two topological spaces is the product of the spaces and so on. The coproduct of two sets is their disjoint union; similarly for topological spaces; the coproduct of two vector spaces is the direct sum; similarly for groups and rings. Note that for groups, rings and vector spaces, the coincidence that the product and coproduct are in fact isomorphic, even though they satisfy two quite different universal properties.

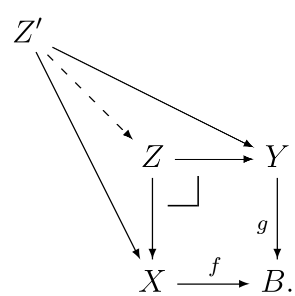
Now let us be a little more ambitious. Take a category with three objects and five morphisms. The two non-trivial morphisms should have the same target, but different domains.

Definition 14.3. *Suppose we are given a diagram*



The limit of the corresponding functor, denoted $X \underset{B}{\times} Y$, is known as the ***fibre product*** *or* ***fibre square****.*

As with the definition of the product, there is an accompanying commutative diagram



Note that if B is a terminal object, then the fibre product is nothing more than the product.

Lemma 14.4. *The category (Sets) of sets admits fibre products.*

Proof. It is easy to check that

$$X \underset{B}{\times} Y = \{ (x, y) \in X \times Y \mid f(x) = g(y) \},$$

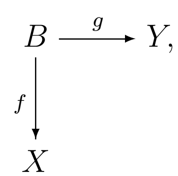
does the trick. ∎

The fibre product is sometimes also known as the pullback. In other words we think of the morphism

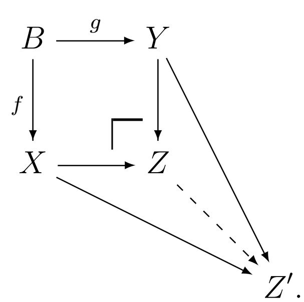
$$X \underset{B}{\times} Y \longrightarrow X,$$

as the pullback of the map $g\colon Y \longrightarrow B$ along the map $f\colon X \longrightarrow B$. In particular the fibre of the former map over the point $x \in X$ is equal to the fibre of the map g over the point $f(x)$.

The dual notion is that of pushout. Basically take the diagram above, flip about the $Y-X$-diagonal and reverse the arrows. Thus if we start with the diagram



the pushout Z enjoys the universal property encoded in the following commutative diagram:



For example, consider the category of rings. Suppose we are given two ring homomorphisms $A \longrightarrow B$ and $A \longrightarrow C$, and two ring homomorphisms $B \longrightarrow P$ and $C \longrightarrow P$. Then we get a bilinear map $B \times C \longrightarrow P$, using multiplication in P. It is then easy to see that the pushout is the tensor product $B \underset{A}{\otimes} C$.