12. Change of coordinates

Definition 12.1. $\mathrm{PGL}_n(K)$ *denotes the space of invertible* $n \times n$ *matrices with entries in* K*, modulo the normal subgroup of scalar matrices, that is*

$$\mathrm{PGL}_n(K) = \frac{\mathrm{GL}_n(K)}{K^*}.$$

Note that the canonical action of $\mathrm{GL}_{n+1}(K)$ on K^{n+1} descends to an action of $\mathrm{GL}_{n+1}(K)$ on $\mathbb{P}^n$, in an obvious way. Clearly the set of scalar matrices acts trivially and in fact it is not hard to see that the scalar matrices are the kernel of the induced homomorphism. On the other hand, it is also easy to see that if we fix a matrix A, then the induced bijection

$$\mathbb{P}^n \longrightarrow \mathbb{P}^n$$

is in fact a morphism. Thus the group $\mathrm{PGL}_n(K)$ is a subgroup of the group of all automorphisms of $\mathbb{P}^n$.

It is interesting to see what happens for $\mathbb{P}^1$. Suppose we take a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then A sends $[X : Y]$ to

$$[aX + bY : cX + dY].$$

Suppose we work in the affine chart $z = X/Y$. Then A sends z to

$$\begin{aligned} \frac{aX + bY}{cX + dY} &= \frac{a(X/Y) + b}{c(X/Y) + d} \\ &= \frac{az + b}{cz + d}. \end{aligned}$$

In the case when $K = \mathbb{C}$, we recover the Möbius group, the group of Möbius transformations.

Perhaps the most interesting property of $\mathrm{PGL}_n(K)$ is the following:

Theorem 12.2. *Let* $p_1, p_2, \dots, p_{n+2}$ *and* $q_1, q_2, \dots, q_{n+2}$ *be two sets of* $n + 2$ *in* $\mathbb{P}^n$ *in linear general position.*

Then there is a unique element of $\phi \in \mathrm{PGL}_n(K)$ *such that*

$$\phi(p_i) = q_i.$$

Using this, in the case $n = 1$, we can give a synthetic construction of the unique conic through five points p_1, p_2, p_3, p_4 and p_5 in linear general position, which is known as the *Steiner construction*. Fix two points $p = p_1$ and $q = p_2$. Consider the set of lines through p.

Definition 12.3. *Suppose that* $\mathbb{P}^n = \mathbb{P}(V)$*. Then* $\hat{\mathbb{P}^n} = \mathbb{P}(V^*)$ *is called the* ***dual projective space****.*

The whole point of $\hat{\mathbb{P}^n}$ is that it parametrises hyperplanes in $\mathbb{P}^n$. Indeed an element of V^* is a linear functional on V. Its zero locus is a hyperplane in V and this defines a hyperplane in $\mathbb{P}^n$. Conversely a hyperplane in $\mathbb{P}^n$ corresponds to a hyperplane in V. This defines a linear functional on V, up to scalars, that is, an element of $\mathbb{P}(V^*)$.

Another way of putting this is as follows. Pick coordinates $X_0, X_1, \dots, X_n$ on V. These form a basis of V^*. A general element of V^* is then of the form

$$a_0X_0 + a_1X_1 + \cdots + a_nX_n,$$

and its zero locus is a hyperplane in $\mathbb{P}^n$.

Lemma 12.4. *Let* $\Lambda \subset \mathbb{P}^n$ *be a linear subspace of* $\mathbb{P}^n$ *of dimension* k*.*

Then the set of linear spaces Γ *of dimension* $k+1$ *(or of dimension* $n-1$*) containing* Λ *is a copy of projective space of dimension* $n-k-1$*.*

Proof. We will give three different proofs of this result and we will also show that these two cases are duals of each other.

The first is geometric. Pick Λ' a complimentary linear subspace (that is, Λ' has the property that it is disjoint from Λ and of maximal dimension with this property). Then Λ' is of dimension $n-k-1$, so that it is a copy of projective space of dimension $n-k-1$.

Claim 12.5. *The points of* Λ' *are in bijection with linear spaces of dimension* $k+1$ *containing* Λ*.*

Proof of (12.5). One direction is clear. Given a point of Λ' the span of this point and Λ is a linear space of dimension $k+1$ containing Λ.

On the other hand, a linear space Γ of dimension $k+1$ containing Λ must meet Λ' in a unique point. Indeed the dimension of the intersection of Γ and Λ' is at least zero. On the other hand, if it were positive dimensional, then there would be a line l in the intersection. This line is contained in Γ and Λ is a hyperplane in Γ, so that l and Λ must meet in a point. But this contradicts the fact that Λ and Λ' are disjoint. $\square$

The second is algebraic. Pick coordinates so that Λ is given as $Z_{k+1} = \dots Z_n = 0$. Then a hyperplane containing Λ is given by an equation of the form

$$a_{k+1}Z_{k+1} + \cdots + a_nZ_n = 0.$$

Thus the set of hyperplanes containing Λ is naturally in bijection with $\mathbb{P}^{n-k-1}$, with coordinates $[a_{k+1} : a_{k+2} : \cdots : a_n]$.

The third uses a little linear algebra. Suppose that $\mathbb{P}^n = \mathbb{P}(V)$. Then V has dimension $n+1$ and $\Lambda = \mathbb{P}(W)$, where W is of dimension

$k+1$. Suppose that $\Gamma = \mathbb{P}(U)$. Then the set of U containing W is in bijection with the set of U' of dimension one in V/W. But the latter is by definition $\mathbb{P}(V/W)$ and as V/W has dimension $n-k$, the result follows. By duality, hyperplanes in $\mathbb{P}(V/W)$ correspond to lines in $\mathbb{P}(V^*/W^*)$ and so the two results are indeed dual. □

Thus the set of lines H_p through p is naturally a copy of $\mathbb{P}^1$. Similarly for the set H_q of lines through q. Choose parametrisations L_t and M_t of these set of lines. Formally we pick isomorphisms $\mathbb{P}^1 \longrightarrow H_p$ and $\mathbb{P}^1 \longrightarrow H_q$. The two lines L_t and M_t intersect in a point p_t. Varying t, the locus of points p_t sweeps out a curve, call it C. First note that C contains p and q, provided that the line $\langle p, q\rangle$ does not correspond to the same parameter value (we will check later that our choice of parametrisations satisfies this condition).

Note that we have three degrees of freedom left. Indeed we may choose our parametrisation of H_p so that $t = 0$ corresponds to the line $\langle p, p_3\rangle$, $t = 1$ to $\langle p, p_4\rangle$ and $t = \infty$ to $\langle p, p_5\rangle$, using (12.2). Similarly for H_q. It follows then that C_t passes through p_3, p_4 and p_5.

It remains to check that C is a conic. There are two ways to see this. The first is by direct computation. If L_t is given by $aX + bY + cZ$ and M_t is given by $dX + eY + fZ$ then the point of intersection of L_t and M_t may be determined as follows. Let A be a square $n \times n$ matrix and let B be the adjugate matrix (the tranpose of the matrix of $n-1 \times n-1$ minors). Then

$$AB = BA = (\det A)I_n.$$

Now let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\det A = 0$ and computing BA we see that the last column of B gives the intersection point of L_t and M_t. So this point has coordinates which are quadratic in a-f. These are in turn linear in S and T, so we get three quadratic polynomials F, G and H. C is then the image of the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2,$$

given by

$$[S : T] \longrightarrow [F : G : H].$$

It is now easy to see that C is a conic:

Lemma 12.6. *Let $C \subset \mathbb{P}^d$ be the image of a morphism*

$$[S : T] \longrightarrow [F_0 : F_1 : \dots : F_d],$$

where $F_0, F_1, \dots, F_d$ have degree at most d.

If C is not contained in a hyperplane then C is projectively equivalent to a rational normal curve of degree d.

Proof. Note that C is not contained in a hyperplane if and only if $F_0, F_1, \dots, F_d$ are linearly independent. Since the space of homogeneous polynomials of degree d in S and T has dimension $d+1$, it follows that the polynomials $F_0, F_1, \dots, F_d$ are a basis for the homogeneous polynomials of degree d. But then we may find a linear transformation taking $S^i T^{d-i}$ to F_i, that is an element of $\mathrm{GL}(d)$, and this defines a projective equivalence with the rational normal curve of degree d. $\square$

Note that in our case if C were to lie in a hyperplane then it would be a line, which is not the case, since C contains p_1, p_2, p_3, p_4 and p_5 and any three of these points are not collinear.

Actually there is another way to check that F, G and H have degree 2. The basic idea is that to find the degree of a curve C, just intersect with a typical line L. The number of points $|C \cap L|$ will just be the degree of the curve. In fact if the line is given by $aX + bY + cZ$ then we just need to find the solutions to the equation

$$aF + bG + cH = 0.$$

If F, G and H have degree d then this equation ought to have d solutions. Borrowing a result from later in the course, in fact we can always choose L with the property that $|L \cap C| = d$ (this is equivalent to saying that not every line is a tangent line).

So pick a typical line L (in particular a line that does not contain any of p_1, p_2, p_3, p_4 and p_5). We calculate $|C \cap L|$.

Since the set L is in (non-canonical) bijection with H_p and H_q, a moments thought will convince the reader that picking L determines an automorphism $\phi \in \mathrm{PGL}(2)$ (indeed send $L_t \cap L$ to $M_t \cap L$) and we want to calculate the number of fixed points of ϕ .

Lemma 12.7. *Let $\phi \in \mathrm{PGL}(2)$.*

Then ϕ is conjugate to

(1) *The identity,*
(2) *$z \longrightarrow az$, some $a \in K^*$,*
(3) *$z \longrightarrow z + 1$.*

Moreover the three cases are determined by the number of fixed points; at least three; two, one.

Thus the degree d of F, G and H is at most 2. If $d = 1$ then C is a line, a contradiction. Thus $d = 2$ and C is a conic.

Now suppose that the line $\langle p, q\rangle$ corresponds to the same parameter value. In this case $d \leq 1$ since one of the fixed points of ϕ corresponds to $L \cap \langle p, q\rangle$, a contradiction. Another way to proceed, which generalises better to higher dimensions, is to consider the line $H = \langle p_3, p_4\rangle$. We get an automorphism of this line, by sending the point $L_t \cap H$ to $M_t \cap H$. This automorphism would have three fixed points, p_3, p_4 and $H \cap \langle p, q\rangle$. But then this automorphism would be the identity. This can only happen if $C = H$ and so p_5 would also lie on H, a contradiction.

This result has the following interesting generalisation:

Theorem 12.8. *Let $p_1, p_2, \dots, p_{n+3}$ be $n+3$ points in linear general position in $\mathbb{P}^n$.*

Then there is a unique rational normal curve through these points.

Proof. We will do the case of $n = 3$ (the general case is no harder, just notationally more involved). Let l be the line $\langle p_1, p_2\rangle$, m be the line $\langle p_2, p_3\rangle$ and n the line $\langle p_1, p_3\rangle$. The set of planes that contains l, H_l, is a copy of $\mathbb{P}^1$. Pick three parametrisations of the three copies of $\mathbb{P}^1$, H_l, H_m and H_n. We choose these parametrisations subject to the condition that the plane spanned by p_1, p_2 and p_3 corresponds to three different parameter values. Given $t \in \mathbb{P}^1$, the three planes corresponding to t intersect in a point, and so we get a curve C in $\mathbb{P}^3$.

Once again we have three degrees of freedom. We may choose our parametrisations, so that $t = 0$ corresponds to the three planes $\langle l, p_4\rangle$, $\langle m, p_4\rangle$ and $\langle n, p_4\rangle$. In this way, we may pick C so that it contains the six points $p_1, p_2, \dots, p_6$ (we check our non-degenerary condition at the end).

It remains to check that C is a twisted cubic. As before we could use the adjugate matrix to conclude that C is the image of

$$[S : T] \longrightarrow [F : G : H : I],$$

where F, G, H and I all have degree three and then we just apply (12.6).

Instead, we could also the geometric argument. As before, it suffices to check that C meets a general plane P in three points. We use the same argument. The planes P_t and Q_t intersect P in a single point x. Similarly the planes Q_t and R_t intersect P in a single point y. H intersects C at the point corresponding to t iff $x = y$. The assignment $x \longrightarrow y$ is an automorphism of P, a copy of $\mathbb{P}^2$, and any automorphism of $\mathbb{P}^2$, not equal to the identity, can have at most three fixed points.

Now suppose that $P_{t_0} = Q_{t_0}$ (necessarily $\langle p_1, p_2, p_3\rangle$), for some t_0. Consider the line $L = \langle p_4, p_5\rangle$. The automorphism given by sending

$P_t \cap L$ to $Q_t \cap L$ would have three fixed points, p_3, p_4 and $L \cap \langle p_1, p_2, p_3 \rangle$. But then p_6 must also lie on L, a contradiction. $\square$