10. Conics in $\mathbb{P}^2$

We want to talk about curves in $\mathbb{P}^2$. For that we need to look at polynomials. The problem is that polynomials in X, Y and Z don't define functions on $\mathbb{P}^2$, since polynomials are not invariant under rescaling. However, we don't really care what the value of the polynomial is, all we care about is whether or not the polynomial is zero.

Definition 10.1. *Let $F(X) \in K[X]$ be a polynomial in the variables $X_0, X_1, \dots, X_n$. We say that F is* ***homogeneous*** *if every non-zero term of F has the same degree d.*

Lemma 10.2. *Let $F(X) \in K[X]$.*

(1) *If F is homogeneous of degree d, then $F(\lambda X) = \lambda^d F(X)$, for all $\lambda \in K$ (we adopt the convention here that $0^0 = 1$).*
(2) *Conversely, if $F(\lambda X) = \lambda^d F(X)$, for all $\lambda \in K$ and K is infinite then F is homogeneous of degree d.*

Proof. (1) is clear. Suppose now that F is any polynomial. Then $F = \sum_i F_i$ has a unique decomposition, where F_i is homogeneous of degree i. If $F(\lambda X) = \lambda^d F(X)$, then this forces $\lambda^i F_i(X) = \lambda^d F_i(X)$, for every i. If K is infinite then for every $i \neq d$, we can pick λ, so that $\lambda^i \neq \lambda^d$. Thus $F_i(X) = 0$, for all $i \neq d$. □

Definition-Lemma 10.3. *Let S be a set of homogeneous polynomials. The zero set $X = V(S)$ is called a* ***projective subvariety*** *of $\mathbb{P}^n$.*

The ***Zariski topology*** *on $\mathbb{P}^n$ is the topology whose closed subsets are the projective subvarieties.*

Lemma 10.4. *$\Lambda \subset \mathbb{P}^n$ is a linear subspace if and only if it is defined by a collection of homogeneous linear equations.*

In particular every linear subspace of $\mathbb{P}^n$ is a projective subvariety.

Proof. Clear, since a subset $W \subset V$ is a linear subspace if and only if it is defined by homogeneous linear equations. □

One of the key points, is that we can go backwards and forwards between affine and projective varieties.

First let us suppose that we are given a subset $V \subset \mathbb{P}^n$. Clearly we can form $V \subset U_0$ simply by intersecting V with U_0. Suppose that V is a closed subvariety, say defined by $F_\alpha(X)$ homogenous. Define $f_\alpha(x)$ by replacing X_i by X_i/X_0. It is pretty easy to see that V_0 is defined by the f_α.

Conversely suppose we are given f_α, which defines $V \subset \mathbb{A}^n$. Then we can form $F_\alpha(X)$ homogeneous, simply by topping up each term of

f_α, by the appropriate power of X_0. This defines $\bar{V}$ in X. Again, it is not hard to see that $\bar{V} \cap U_0 = V$.

In both cases, the best way to see what is going on, is to look at some examples.

Example 10.5. *Suppose we consider $x^2 + y^2 = 1$ inside $\mathbb{A}^2$. We may think of this as $U_2 \subset \mathbb{P}^3$, with coordinates X, Y and Z. Replace $x^2+y^2 = 1$ by x^2+y^2-1. This has degree two. The first two terms have degree two, and there is nothing to do (apart from replacing lower caps by upper). The last term has degree zero. To make this homogeneous then, we need to multiply by Z^2. We get $X^2 + Y^2 - Z^2$. Now suppose we want to work on U_0. Then we divide through by X^2 and replace Y/X by y and Z/X by z, to get $1 + y^2 - z^2$. Note the quick way to do this is simply to replace X by 1 and replace upper caps by lower.*

Example 10.6. *Consider $y = x^3$. We get $y - x^3$. Consider this inside $\mathbb{P}^2$, with coordinates X, Y and Z. We get $YZ^2 - X^3$. Now work inside U_1. We get $z^2 - x^3$.*

It is interesting to see what happens to parallel lines in $\mathbb{A}^2$.

Example 10.7. *Let L be the line $y = 0$ and let M_t be the line $y = tx + 1$, where $t \in K$. Then L becomes the line $Y = 0$ and M_t the line $Y = tX + Z$. When $t = 0$, we get $Y = Z$. Thus $Z = 0$, and we get the point $[1 : 0 : 0]$. Thus our two parallel lines intersect along the line at infinity, at the point $[1 : 0 : 0]$, corresponding to the fact that both lines are horizontal.*

In fact it is interesting to consider the family in the coordinate patch $Y \neq 0$. We get $x = 0$ and $x = t + z$, which is equivalent to $x = 0$ and $z = -t$.

Note that these processes are not quite inverse.

Example 10.8. *Suppose we start with $X = 0$ inside $\mathbb{P}^2$. If we go to the coordinate patch U_0 then we get the empty set. Going back to $\mathbb{P}^2$, we get the empty set. The whole point is that the whole of $X = 0$ completely avoids the set U_0.*

One of the beautiful results of classical projective geometry is the following:

Lemma 10.9. *Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree two. Suppose that $f = 0$ contains more than one real point. Let F be the homogenisation of f.*

Then $f = 0$ is a circle if and only if $F = 0$ contains the points $[1 : \pm i : 0]$.

Proof. Suppose that $f = 0$ defines a circle. Then $f(x, y)$ has the form

$$(x - a)^2 + (y - b)^2 = r^2.$$

Thus F is equal to

$$(X - aZ)^2 + (Y - bZ)^2 = r^2 Z^2.$$

Set $Z = 0$. Then $X^2 + Y^2 = 0$, which has the solution $[1 : \pm i : 0]$.

Conversely suppose that $F = 0$ contains the points $[1 : \pm i : 0]$. Then

$$F(X, Y, 0) = aX^2 + bXY + cY^2,$$

vanishes at $[1 : \pm i : 0]$. Thus

$$ax^2 + bx + c = 0,$$

has roots $\pm i$, which is only possible if $b = 0$ and $a = c$. Hence $F(X, Y, 0)$ is a non-zero multiple of $X^2 + Y^2$. Possibly rescaling, we may assume that

$$F(X, Y, Z) = X^2 + Y^2 + ZG(X, Y, Z)$$

where $G(X, Y, Z)$ is a linear polynomial. Thus

$$f(x, y) = x^2 + y^2 + g(x, y),$$

for some linear polynomial g. Completing the square, we can put this in the form

$$(x - a)^2 + (y - b)^2 = k.$$

The condition that $f = 0$ contains more than one point is equivalent to requiring that $k > 0$, so that $k = r^2$, some $r > 0$ and we have the equation of a circle. □

Since we want to work over $\mathbb{C}$, it turns out that we want to reinvent the wheel:

Definition 10.10. *The curve $C \subset \mathbb{P}^2_{\mathbb{C}}$, given as $F = 0$, is a* ***circle*** *if F has degree two and C contains the points $[1 : \pm i : 0]$.*

Let us consider the general polynomial of degree two in X, Y and Z,

$$F(X, Y, Z) = aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY,$$

where a, b, c, d, e and f are in K. Thus polynomials of degree two are naturally in correspondence with K^6. On the other, if $F = \lambda G$, $\lambda \neq 0$, then F and G define the same zero locus. Over an algebraically closed field, the converse is true. Thus the set of conics in $\mathbb{P}^2$ is naturally in bijection with K^6 modulo scalars, that is, $\mathbb{P}^5$.

Given that we want to count how many circles pass through two points and that a circle is nothing more than a conic that passes through two fixed points, the natural problem is to identify the following locus:

$$H_p = \{\, [a : b : c : d : e : f] \in \mathbb{P}^5 \mid F = 0 \text{ passes through } p \,\},$$

where $p \in \mathbb{P}^2$ is a point.

Lemma 10.11. *$H_p \subset \mathbb{P}^5$ is a hyperplane (that is, a linear space defined by a single equation).*

Proof. Indeed, if $p = [u : v : w]$, then $[a : b : c : d : e : f] \in H_p$ if and only if $[a : b : c : d : e : f]$ satisfies the linear equation

$$u^2A + v^2B + w^2C + (vw)D + (uw)E + (uv)F = 0. \qquad \square$$

For example, the conic passes through $p = [0 : 0 : 1]$ if and only if the coefficient of Z^2 is zero if and only if $c = 0$.

Lemma 10.12. *Suppose we are given five points p_1, p_2, p_3, p_4 and p_5, and we are working over an infinite field.*

Then, either there is a unique conic through these points, or infinitely many.

Proof. Let $H_i \subset \mathbb{P}^5$ be the hyperplane corresponding to p_i. Then the set of conics passing through the given points corresponds to the intersection of the five hyperplanes. As the intersection of linear spaces is a linear space, the result follows. $\square$

Theorem 10.13. *There is a unique conic passing through five points in linear general position.*

Proof. Suppose not. Then the intersection of the five hyperplanes H_1, H_2, H_3, H_4 and H_5 would contain a line, call it $l \subset \mathbb{P}^5$. Pick two points of this line, corresponding to two quadratic polynomials F and G. As any two points on l, span l, the general point of l is given as $[sF + tG]$, for $[s : t] \in \mathbb{P}^1$. Thus the curve $sF + tG = 0$ contains the five given points p_1, p_2, p_3, p_4 and p_5.

Pick any point $p \in \mathbb{P}^2$. Then we may find $[s : t] \in \mathbb{P}^1$ such that $(sF + tG)(p) = 0$. Indeed, if $G(p) = 0$, take $[s : t] = [0 : 1]$, else set $s = 1$ and

$$t = -\frac{F(p)}{G(p)}.$$

Now pick p collinear with p_1 and p_2. Then the conic C corresponding to $sF + tG = 0$ contains the three points p_1, p_2 and p of the line $m = \langle p_1, p_2 \rangle$. Pick coordinates so that m is given as $Z = 0$. Then the quadratic polynomial

$$F(X, Y, 0),$$

has three zeroes. It follows that $F(X, Y, 0) = 0$, so that $F(X, Y, Z) = ZG(X, Y, Z)$. In other words the curve C is the union of the two lines $Z = 0$ and $G = 0$. But then one of the two lines contains three of our five points, which contradicts our assumption that the points are in linear general position. $\square$

Corollary 10.14. *There is a unique circle passing through three non-collinear points in* $\mathbb{R}^2$.

Proof. Note that the line spanned by the points $[1 : \pm i : 0]$ is the line at infinity of $\mathbb{P}^2_{\mathbb{C}}$. Thus given three points p, q and r in $\mathbb{R}^2$, which are not collinear, then the five points p, q, r and $[1 : \pm i : 0]$ are in linear general position in $\mathbb{P}^2_{\mathbb{C}}$.

By (10.13) there is a unique conic through the five given points. Now of the five hyperplanes that define this conic, three are defined by linear equations with real coefficients and even though the other two have complex roots, the equations of the hyperplanes are complex conjugates. Since the set of solutions to a set of equations which is invariant under complex conjugation, is invariant under complex conjugation, it follows that this unique solution has coefficients which are invariant under complex conjugation, which is to say that it is a point with real coordinates. In particular the definining equation of the unique conic passing through the five given points is real. On the other hand the corresponding curve contains three real points. Therefore by (10.9) there is a unique circle through the three real points. $\square$

Note the fancy footwork needed to deal with the problem of working over non algebraically closed fields.

It turns out there are other ways to prove (10.13).