Math 20E Homework Assignment 6 ***Updated May 29***
Due 11:00pm Thursday, June 6, 2024

1. Use the divergence theorem to calculate the flux of $\mathbf{F} = (x-y)\,\mathbf{i} + (y-z)\,\mathbf{j} + (z-x)\,\mathbf{k}$ out of the unit sphere $x^2 + y^2 + z^2 = 1$.

2. Let S be the boundary surface of a solid region W. Show that

$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = 3 \text{ volume}(W).$$

   Explain this result geometrically.

3. Let W be the pyramid with top vertex $(0,0,1)$, and base vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,0)$. Let S be the closed boundary surface of W, oriented outward from W. Let $\mathbf{F}(x,y,z) = (x^2y,\ 3y^2z,\ 9z^2x)$. Use Gauss' theorem to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

4. Let $\mathbf{F}(x,y,z) = (x+y,\ z,\ z-x)$ and let $\mathcal{S}$ be the boundary surface of the solid region between the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

5. Let W be a symmetric elementary region in $\mathbb{R}^3$ with positively oriented closed boundary surface ∂W. Show that

$$\iiint_W \frac{1}{r^2}\, dx\, dy\, dz = \iint_{\partial W} \frac{1}{r^2} \mathbf{r} \cdot \mathbf{n}\, dS,$$

   where $\mathbf{r} = (x,y,z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$.

6. Let $\mathbf{r} = (x,y,z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2+y^2+z^2}$. Verify the following identities.
   (a) $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\mathbf{r}$ for $r \neq 0$; and, in general, $\nabla(r^n) = nr^{n-2}\,\mathbf{r}$, and $\nabla \log(r) = \frac{1}{r^2}\,\mathbf{r}$.
   (b) $\nabla^2\left(\frac{1}{r}\right) = 0$ for $r \neq 0$; and, in general, $\nabla^2 r^n = n(n+1)r^{n-2}$.
   (c) $\boldsymbol{\nabla} \cdot \left(\frac{1}{r^3}\,\mathbf{r}\right) = 0$; and, in general, $\boldsymbol{\nabla} \cdot (r^n\,\mathbf{r}) = (n+3)r^n$
   (d) $\boldsymbol{\nabla} \times \mathbf{r} = \mathbf{0}$; and, in general, $\boldsymbol{\nabla} \times (r^n\,\mathbf{r}) = \mathbf{0}$.

7. Let $\mathbf{F}(x,y,z) = y\mathbf{i} + [z\cos(yz) + x]\mathbf{j} + y\cos(yz)\mathbf{k}$.
   (a) Verify that $\mathbf{F}$ is irrotational.
   (b) Find a scalar potential for $\mathbf{F}$.

8. Let $\mathbf{F}(x,y,z) = [2xyz + \sin(x)]\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$.
   Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}(t) = (\cos^5(t),\ \sin^3(t),\ t^4)$ for $0 \le t \le \pi$.