Math 142B Homework Assignment 5
Due 11:00pm Thursday, June 6, 2024

1. Let $\sum a_n x^n$ be a power series with radius of convergence R. Prove:
   (a) If all the coefficients a_n are integers and $a_n \neq 0$ for infinitely many n, then $R \leq 1$.
   (b) If $\limsup |a_n| > 0$, then $R \leq 1$.

2. (a) Suppose $\sum a_n x^n$ has finite radius of convergence R and $a_n \geq 0$ for all n. Show that if the series converges at $x = R$, then it also converges at $x = -R$.
   (b) Exhibit an example of a power series whose interval of convergence is exactly $(-1, 1]$. (**Note:** "Exhibit" means "Show that the example has the required properties.")

3. (a) Verify that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ for all $x \in \mathbb{R}$, since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$.
   (b) Write $F(x) = \int_0^x e^{-t^2}\, dt$ as a power series. Be sure to briefly explain how you know that the power series for $F(x)$ converges for all $x \in \mathbb{R}$.

4. Let $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for $x \in \mathbb{R}$. Using only the properties of power series, show that $f' = f$.

5. For $x \in \mathbb{R}$, let

$$s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$c(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

   Prove:
   (a) $s' = c$ and $c' = -s$.
   (b) $\left(s^2 + c^2\right)' = 0$.
   (c) $s^2 + c^2 = 1$.

6. (a) Show that $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$ for $x \in (-1, 1)$.
   (b) Show that $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $x \in (-1, 1)$.
   (c) Show that the equality in (b) also holds for $x = 1$. Use this to find a fun formula for π.
   (d) What happens at $x = -1$?

7. Find the Taylor series for $\cos(x)$ and indicate why it converges to $\cos(x)$ for all $x \in \mathbb{R}$.

8. Let $g(x) = \begin{cases} 0 & \text{if } x = 0, \\ e^{-1/x^2} & \text{otherwise.} \end{cases}$
   (a) Show that $g^{(n)}(0)$ for all $n \in \mathbb{N}$.
   (b) Show that the Taylor series for g about 0 agrees with g only at $x = 0$.

9. Prove that $|\sin(x+h) - (\sin(x) + h\cos(x))| \leq \dfrac{h^2}{2}$ for every pair of real numbers x and h.

10. Suppose f is differentiable on (a,b), f' is bounded on (a,b), f' never vanishes on (a,b), and the sequence (x_n) in (a,b) converges to $\bar{x} \in (a,b)$.

    Show that if $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$ for all $n \geq 0$, then $f(\bar{x}) = 0$.