Math 142A Homework Assignment 5
Due Tuesday, March 12, 2024

1. Prove that a polynomial $p(x)$ with odd degree has a least one real zero.

2. Suppose that the function $f : [a,\ b] \to \mathbb{R}$ is continuous. Show that for any $n \in \mathbb{N}$ and points $x_1, x_n, \ldots, x_n \in [a,\ b]$, there is a point $z \in [a,\ b]$ such that $f(z) = \dfrac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$.

3. Suppose that the function $f : [0,\ 1] \to \mathbb{R}$ is continuous with $f(0) > 0$, and $f(1) = 0$. Show that there is a number $x_0 \in (0,\ 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for every $x \in [0,\ x_0)$.

4. Let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$

   Show that f has the intermediate value property on all of $\mathbb{R}$.

5. Suppose that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is periodic; that is, there is a number $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

6. Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

7. Let f be a continuous function on $[0,\ \infty)$. Prove that if f is uniformly continuous on $[k,\ \infty)$ for some $k > 0$, then f is uniformly continuous on $[0,\ \infty)$.

8. Let f be a continuous function on $[a,\ b]$. Show that the function f^* defined by

   $$f^*(x) = \sup\left\{f(y) \mid a \le y \le x\right\}, \text{ for } x \in [a,\ b],$$

   is an increasing continuous function on $[a,\ b]$.

9. Suppose the limits $L_1 = \lim\limits_{x \to a^+} f_1(x)$ and $L_2 = \lim\limits_{x \to a^+} f_2(x)$ both exist.
   (a) Show that if $f_1(x) \le f_2(x)$ for all x in some interval $(a,\ b)$, then $L_1 \le L_2$.
   (b) Suppose $f_1(x) < f_2(x)$ for all x in some interval $(a,\ b)$. Can you conclude that $L_1 < L_2$?

10. Show that if $\lim\limits_{x \to a^+} f_1(x) = \lim\limits_{x \to a^+} f_3(x) = L$ and if $f_1(x) \le f_2(x) \le f_3(x)$ for all x in some interval $(a,\ b)$, then $\lim\limits_{x \to a^+} f_2(x) = L$. (Note: Be sure to show that $\lim\limits_{x \to a^+} f_2(x)$ exists, since this is not assumed.)