1. Let \((s_n)\) and \((t_n)\) be bounded sequences of nonnegative real numbers. Prove that
\[\limsup s_n t_n \leq (\limsup s_n) (\limsup t_n).\]

2. Prove that \((s_n)\) is bounded if and only if \(\limsup |s_n| \in \mathbb{R}\) (that is, \(\limsup |s_n| < +\infty\)).

3. Let \((s_n)\) be a bounded sequence of nonzero real numbers. Prove that
\[\liminf \frac{s_{n+1}}{s_n} \leq \liminf |s_n|^{1/n}.\]

4. Let \((s_n)\) be a sequence of nonnegative numbers. For each \(n\), define \(\sigma_n = \frac{1}{n} (s_1 + s_2 + \cdots + s_n)\). [Recall from Homework 2 that \((\sigma_n)\) is called the sequence of Cesàro means for \((s_n)\).]
   (a) Show that \(\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n\).
   (b) Show that if \(\lim s_n\) exists, then \(\lim \sigma_n\) exists and \(\lim \sigma_n = \lim s_n\).
   (c) Exhibit an example for which \(\lim \sigma_n\) exists, but \(\lim s_n\) does not exist.

5. Show that if \(\sum a_n\) and \(\sum b_n\) are convergent series of nonnegative numbers, then \(\sum \sqrt{a_n b_n}\) converges.

6. Find a series \(\sum a_n\) which diverges by the Root Test but for which the Ratio Test gives no information.

7. Let \((a_n)\) be a sequence of nonzero real numbers such that the sequence \(\left(\frac{a_{n+1}}{a_n}\right)\) is a constant sequence. Show that \(\sum a_n\) is a geometric series.

8. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence such that \(\liminf |a_n| = 0\). Prove there is a subsequence \((a_{n_k})_{k \in \mathbb{N}}\) such that \(\sum_{k=1}^{\infty} a_{n_k}\) converges.

9. (a) Exhibit an example of a divergent series \(\sum a_n\) for which \(\sum a_n^2\) converges.
   (b) Show that if \(\sum a_n\) is a convergent series of nonnegative terms, then \(\sum a_n^2\) also converges.
   (c) Exhibit an example of a convergent series \(\sum a_n\) for which \(\sum a_n^2\) diverges.

10. Prove that if \((a_n)\) is a decreasing sequence of positive real numbers and if \(\sum a_n\) converges, then \(\lim_{n \to \infty} na_n = 0\).