1. Let \( b \) be a nonzero integer and let \( a, q \) and \( r \) be integers such that \( a = bq + r \). Prove that \( \gcd(a, b) = \gcd(b, r) \).

2. Let \( n \) be a positive integer and let \( a \) be an integer coprime to \( n \). Prove that for every integer \( b \), there is an integer \( x \) such that \( ax \) is divisible by \( n \).

3. Let \( a, b \) and \( c \) be integers such that \( \gcd(a, c) = \gcd(b, c) = 1 \). Prove that \( \gcd(ab, c) = 1 \).

4. Let \( a, b \) and \( c \) be integers such that \( a \) and \( b \) are coprime and \( c \) divides \( a + b \). Prove that \( \gcd(a, c) = \gcd(b, c) = 1 \).

5. Show that \( \gcd(5n + 2, 12n + 5) = 1 \) for every integer \( n \).

6. Let \( p \) and \( q \) be integers such that 3 divides \( p^2 + q^2 \). Prove that 3 divides \( p \) and 3 divides \( q \).

7. Find a positive integer \( n \) and members \([a] \) and \([b] \) of \( \mathbb{Z}_n \) such that \([a] \cdot [b] = [0] \) but \([a] \neq [0] \) and \([b] \neq [0] \).

8. Prove that the nonzero element \([a] \) of \( \mathbb{Z}_n \) has a multiplicative inverse in \( \mathbb{Z}_n \) if and only if \( n \) and \( a \) are coprime.

9. Define \( \simeq \) on \( \mathbb{R} \) by \( x \simeq y \) if and only if \( x - y \) is an integer.
   (a) Prove that \( \simeq \) is an equivalence relation on \( \mathbb{R} \).
   (b) Which real numbers belong to \([−17] \)?
   (c) Characterize the partition \( \Pi \) on \( \mathbb{R} \) corresponding to \( \simeq \).

10. Define \( \sim \) on the set \( M_{n \times n} \) of all \( n \times n \) matrices by \( A \sim B \) if and only if there is an invertible matrix \( P \in M_{n \times n} \) such that \( B = P^{-1}AP \). Prove that \( \sim \) is an equivalence relation on \( M_{n \times n} \).

11. For each real number \( b \), let \( A_b = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = |x + b| \} \), and let \( \mathcal{A} = \{A_b \mid b \in \mathbb{R} \} \). Is \( \mathcal{A} \) a partition of \( \mathbb{R} \times \mathbb{R} \)? Justify your answer.

12. For each real number \( b \), let \( A_b = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = |x| + b \} \), and let \( \mathcal{A} = \{A_b \mid b \in \mathbb{R} \} \). Is \( \mathcal{A} \) a partition of \( \mathbb{R} \times \mathbb{R} \)? Justify your answer.

13. Let \( f : A \to A \) be a function from a set \( A \) to itself.
   (a) Given that \( A \) is finite, prove that \( f \) is injective if and only if \( f \) is surjective.
   (b) Let \( A = \mathbb{Z}^+ \). Find a function \( f_1 : A \to A \) that is injective but not surjective, and find a function \( f_2 : A \to A \) that is surjective but not injective.

14. Since \((0, 1)\) and \([0, 1]\) have the same cardinality, there must be a bijection \( \sigma : (0, 1) \to [0, 1] \) between them. Find an explicit formula for one.

15. Let \( \mathbb{N} = \{n \in \mathbb{Z} \mid n \geq 0\} \). (\( \mathbb{N} \) is often called the set of natural numbers.) Let \( \mathcal{F}(\mathbb{N}) \) be the collection of all finite subsets of \( \mathbb{N} \). Find an explicit bijection \( \sigma : \mathcal{F}(\mathbb{N}) \to \mathbb{N} \). (Hint: Think about binary representation of integers.)