The Compensating Polar Planimeter

- Description of a polar planimeter
- Standard operation
- The neutral circle
- How a compensating polar planimeter compensates
- Show and tell: actual planimeters

References (Far from exhaustive!)

1. Henrici, O., Report on Planimeters, *Report of the Sixty-Fourth Meeting of the British Association for the Advancement of Science (1894)* pp. 496-523.

2. Leise, Tanya, As the Planimeter's Wheel Turns: Planimeter Proofs for Calculus Class, *The College Mathematics Journal*, Vol. 38, No. 1, January 2007, pp. 24-31.

3. Casselman, Bill and Eggers, John, The Mathematics of Surveying: Part II. The Planimeter, *The Mathematics of Surveying: Part II. The Planimeter*, American Mathematical Society Feature Column, June-July 2008

4. Los Angeles Scientific Instrument Company (LASICO)

   Planimeter models, http://www.lasico.com/planimeters.htm

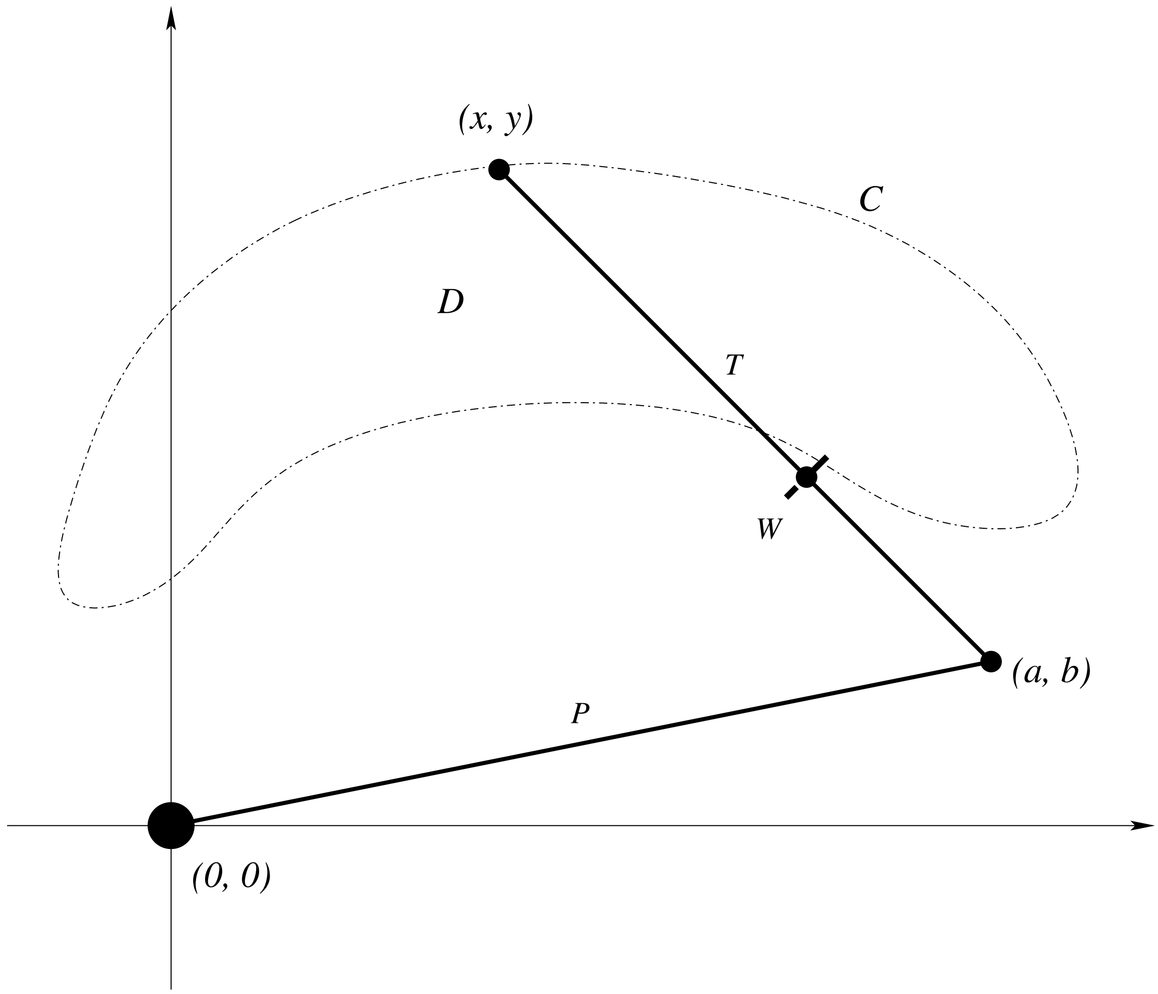
Online Resources

1. Clark McCoy's Keuffel & Esser Catalog Site
   http://www.mccoys-kecatalogs.com/

2. John Eggers' Planimeter Gallery
   http://www.math.ucsd.edu/ jeggers/Planimeter/

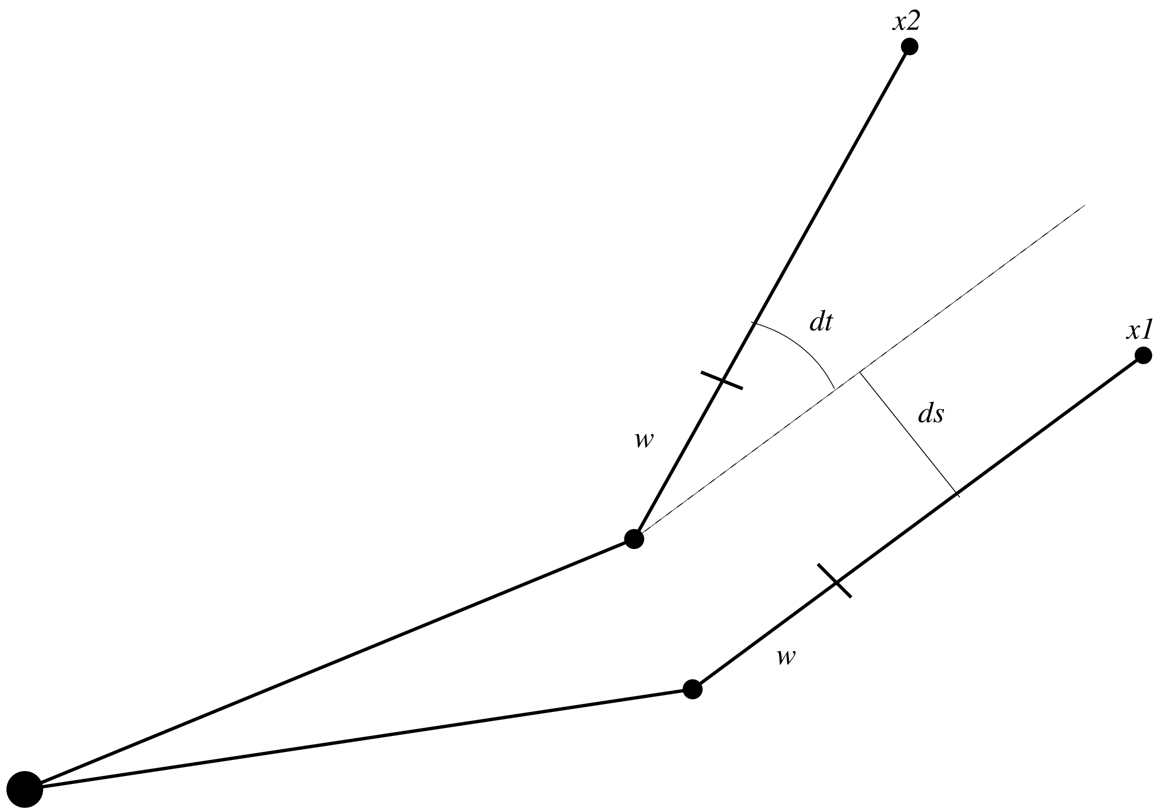
3. and many others...

The polar planimeter was invented in 1854 by Jakob Amsler.



A Polar Planimeter

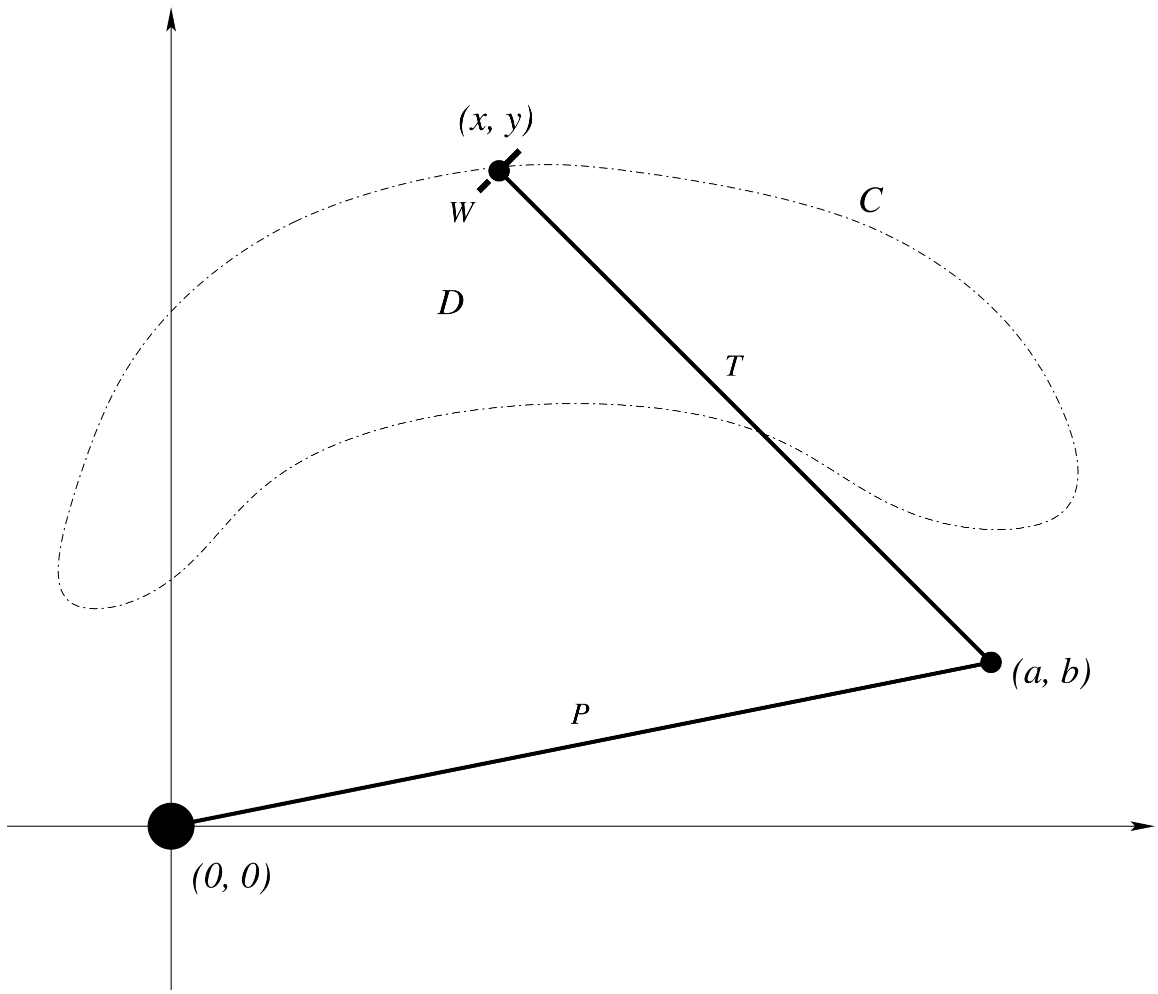
In fact, the measuring wheel W can be placed anywhere along the tracer arm T.



A Small Planimeter Motion

The total wheel displacement from a tracing operation is $\int_C ds + w \int_C d\theta$.

Since $\int_C d\theta = 0$ around a closed curve C, the wheel displacement is $\int_C ds$, which is independent of the position w of the wheel.



An Idealized Polar Planimeter

We will denote by Ω and $\partial\Omega$ the disk and circle of radius $P+T$, respectively, centered at $(0,0)$. They represent the maximal region and boundary curve accessible to the planimeter. We will call a domain D whose closure is contained in Ω an *accessible* domain.

Theorem (Green's Theorem)**.** Let D be a simply connected domain bounded by a simple closed curve C. Let $P(x,y)$ and $Q(x,y)$ be C^1 on a neighborhood of $D \cup C$. Then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Measuring wheel displacement: $M = \int_C \boldsymbol{\tau} \cdot d\mathbf{r}$.

$\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y-b\right), x-a\right)$ the positively-oriented unit vector perpendicular to the tracer arm.

Thus, $M = \dfrac{1}{T}\displaystyle\int_C -\left(y-b\right)\,dx + \left(x-a\right)\,dy$.

In practice, planimeters are manufactured so that positive wheel displacement corresponds to a *clockwise* traversal of C, but we will stay with the standard mathematical convention to avoid confusion.

The coordinates (a, b) are functions of the coordinates (x, y). Applying Green's Theorem to the line integral representing M yields

$$M = \frac{1}{T} \iint_D \left[2 - \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \right) \right] dx\, dy.$$

If $\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = 1$ on D, it will follow that the displacement M of the measuring wheel is proportional to the area $A(D) = \iint_D dx\, dy$.

The coordinates (a, b) satisfy

$$a^2 + b^2 = P^2$$
$$(x - a)^2 + (y - b)^2 = T^2$$

where P and T are the length of the pole arm and tracer arm, respectively. Thus, (a, b) satisfy the following system of partial differential equations:

$$(1) \quad a\,\frac{\partial a}{\partial x} + b\,\frac{\partial b}{\partial x} = 0 \qquad (2) \quad a\,\frac{\partial a}{\partial y} + b\,\frac{\partial b}{\partial y} = 0$$
$$(3) \quad x\,\frac{\partial a}{\partial x} + y\,\frac{\partial b}{\partial x} = x - a \qquad (4) \quad x\,\frac{\partial a}{\partial y} + y\,\frac{\partial b}{\partial y} = y - b$$

Treating this as a system of four linear equations in the four unknowns $\frac{\partial a}{\partial x}$, $\frac{\partial a}{\partial y}$, $\frac{\partial b}{\partial x}$ and $\frac{\partial b}{\partial y}$, we find that

$$\frac{\partial a}{\partial x} = -\frac{b(x-a)}{ay-bx} \qquad \frac{\partial a}{\partial y} = -\frac{b(y-b)}{ay-bx}$$
$$\frac{\partial b}{\partial x} = \frac{a(x-a)}{ay-bx} \qquad \frac{\partial b}{\partial y} = \frac{a(y-b)}{ay-bx}$$

Therefore,

$$\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} = \frac{ay-bx}{ay-bx} = 1,$$

provided

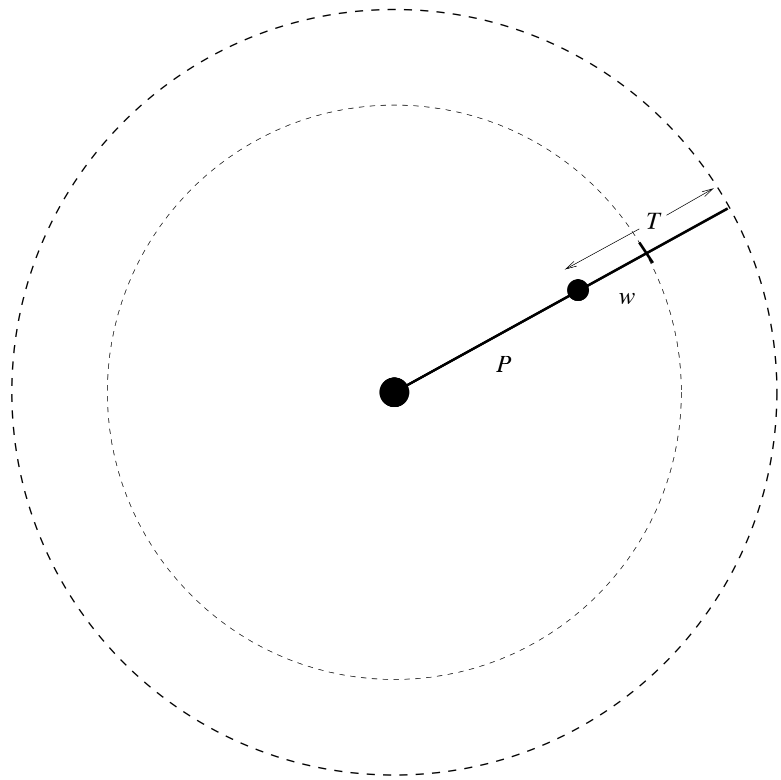
$$(x,y) \neq \lambda(a,b) \quad \text{for any constant } \lambda.$$

Note that $(x,y) = \lambda(a,b)$ only if $(x,y) \in \partial\Omega$ or $(x,y) = (0,0)$.

This means that if D is an accessible domain and $(0, 0) \notin D$, then the planimeter vector field $\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y-b\right), x-a\right)$ is continuously differentiable on $D \cup C$ and it follows from Green's theorem that

$$M = \frac{1}{T}\iint_D dx\, dy = \frac{1}{T}A(D).$$

Green's theorem cannot be applied directly when the pole $(0,0)$ is in the domain D. We first consider the special case of the disk Ω of radius $R = P + T$ centered at the pole $(0,0)$.

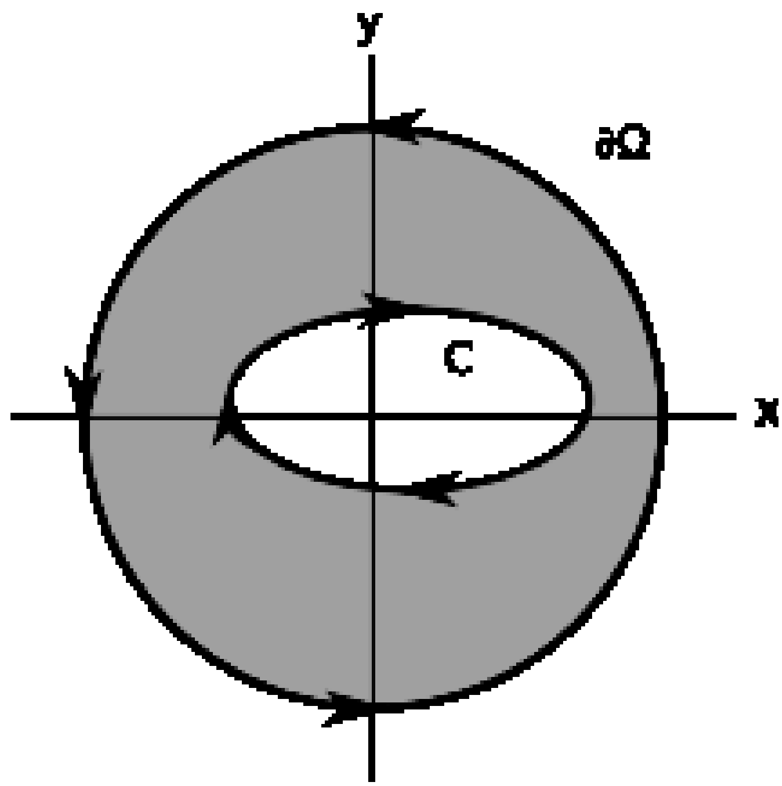


M and $A(\Omega)$ can be computed directly:

$$M_\Omega = 2\pi(P + w)$$
$$A(\Omega) = \pi(P + T)^2$$

Thus, $A(\Omega) = TM_\Omega + \pi\left[(P - T)^2 + 2T(P - w)\right]$.

Next, we consider a simple closed curve C interior to Ω and enclosing $(0,0)$.



By Green's Theorem, the area between C and $\partial\Omega$ can be measured directly by the planimeter: $A(\Omega) - A(C) = T\,M_\Omega - T\,M_C$. Thus, $A(C) = T\,M_C + \pi\left[(P-T)^2 + 2\,T\,(P-w)\right]$ for every simple closed curve C containing $(0,0)$ that is within the planimeter's reach.

$w < T \leq P$ for most polar planimeters. Thus, $\pi\left[(P-T)^2 + 2T(P-w)\right] > 0$ and is the area of a circle of radius $\sqrt{(P-T)^2 + 2T(P-w)}$.

The circle N of radius $\sqrt{(P-T)^2 + 2T(P-w)}$ and centered at the pole $(0,0)$ is called the *neutral circle* since tracing this circle with the planimeter would result in a total wheel displacement $M_N = 0$.

Most planimeter manufacturers would test each instrument they produced to determine the area of the neutral circle and would include this information with the instrument.

In a compensating polar planimeter, the pole and tracer arms are separate pieces that fit together via a ball-and-socket pivot joint.

The design allows the instrument to be set up in two distinct orientations with the pivot joint on either side of the line through the pole and tracer point.

By taking readings with each orientation of the pivot joint and averaging the results, errors caused by misalignment of the measuring wheel exactly cancel; thus, the design allows one to *compensate* for this type of error.

$\boldsymbol{\tau} = \frac{1}{T}\left(-\left(y-b\right), x-a\right)$ positively-oriented unit vector perpendicular to the tracer arm

$\boldsymbol{\rho} = \frac{1}{T}(x-a, y-b)$ positively-oriented (outward) unit vector parallel to the tracer arm

If measuring wheel axis is misaligned by angle ϑ, the unit vector $\boldsymbol{w}$ in direction of $(+)$ wheel displacement is no longer $\perp$ to tracer arm:

$\boldsymbol{w} = \cos(\vartheta)\boldsymbol{\tau} + \sin(\vartheta)\boldsymbol{\rho}$.

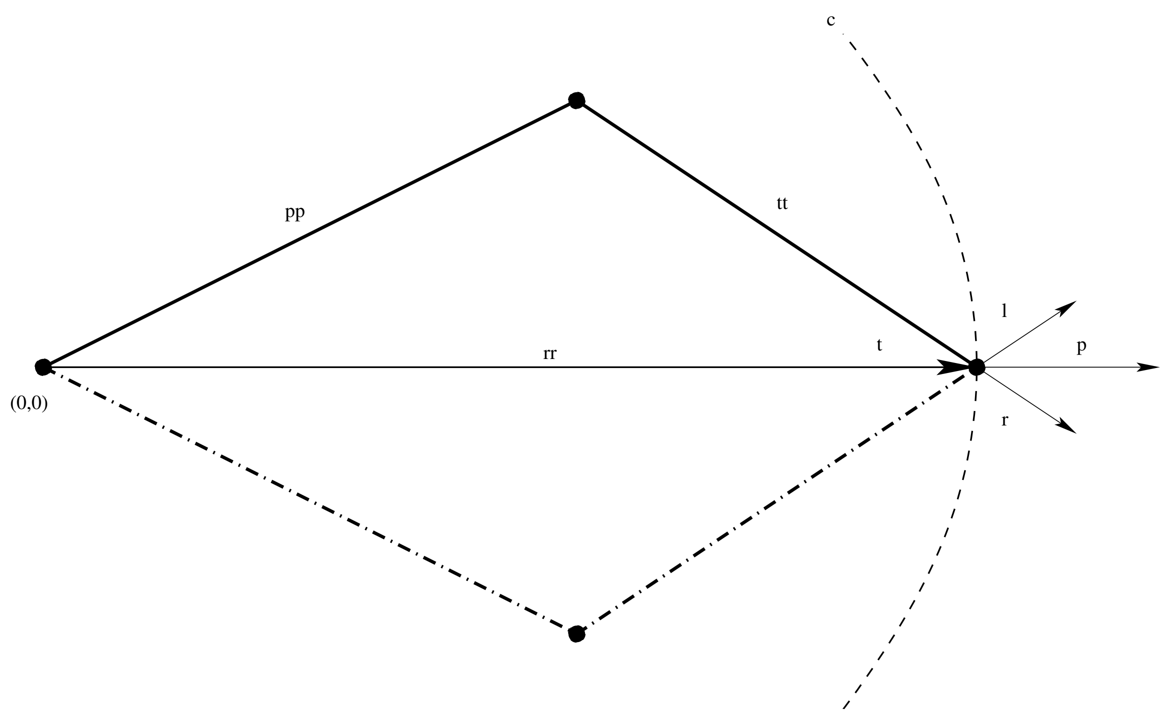
The displacement M after tracing a curve C enclosing a domain D is

$$\begin{aligned} M &= \cos(\vartheta)\int_C \boldsymbol{\tau}\cdot d\mathbf{r} + \sin(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r} \\ &= \frac{\cos(\vartheta)}{T}A(D) + \sin(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r}. \end{aligned}$$

In other words,

$$A(D) = T\,\sec(\vartheta)\,M - T\tan(\vartheta)\int_C \boldsymbol{\rho}\cdot d\mathbf{r}.$$

If the compensating polar planimeter is placed in the two possible configurations, at any point $\boldsymbol{r} = (x, y)$ on the curve C, there are two possible values for $\boldsymbol{\rho}$: $\boldsymbol{\rho}_L$ and $\boldsymbol{\rho}_R$ (see figure).



The sum of the unit vectors $\boldsymbol{\rho}_L$ and $\boldsymbol{\rho}_R$ is parallel to the vector $\boldsymbol{r} = (x, y)$.

The length of $\boldsymbol{\rho}_L + \boldsymbol{\rho}_R$ depends only on the distance $r = \sqrt{x^2 + y^2}$ between the pole $(0, 0)$ and the tracer point (x, y).

Thus, $\boldsymbol{\rho}_L + \boldsymbol{\rho}_R = f(r)\boldsymbol{r}$, where $\boldsymbol{r} = (x, y)$ and $r = \sqrt{x^2 + y^2}$.

Since $\boldsymbol{\nabla} \times f(r)\boldsymbol{r} = \mathbf{0}$, it follows that

$$\int_C f(r)\mathbf{r} \cdot d\boldsymbol{r} = 0$$

around any simple closed curve C.

Hence,

$$\int_C \boldsymbol{\rho}_L \cdot d\boldsymbol{r} + \int_C \boldsymbol{\rho}_R \cdot d\boldsymbol{r} = 0$$

for every simple closed curve.

Thus averaging readings of the compensating polar planimeter taken with the two configurations eliminates error due to misalignment of measuring wheel.

The design of the compensating polar planimeters allows more precise measurement.

By the 1930's, compensating polar planimeters had essentially displaced the original Amsler design.