Math 120A **Homework 5** **Due September 5, 2024**

1. Suppose that z_0 is an isolated singularity of $f(z)$ and that $(z-z_0)^N f(z)$ is bounded near z_0. Show that z_0 is either removable or a pole of order at most N.

2. Suppose that z_0 is an isolated singularity of $f(z)$ that is not removable. Show that z_0 is an essential singularity of $e^{f(z)}$.

3. Obtain the partial fraction decomposition of $f(z) = \dfrac{z^6}{(z^2+1)(z-1)^2}$.

4. $g(z) = e^{\frac{1}{z}}$ has an essential singularity at 0.
   (a) Compute the residue of $g(z)$ at 0.
   (b) Evaluate the integral $\displaystyle\oint_{|z|=2} g(z)\,dz$.

5. Evaluate $\displaystyle\oint_{|z|=2} \frac{z}{\cos(z)}\,dz$.

6. Suppose $P(z)$ and $Q(z)$ are polynomials with $\deg(P) < \deg(Q)$ and such that the zeros of $Q(z)$ are simple zeros at the points $z_1, \ldots, z_N$. Show that the partial fraction decomposition of $f(z) = \dfrac{P(z)}{Q(z)}$ is given by
$$f(z) = \sum_{k=1}^{N} \frac{P(z_k)}{Q'(z_k)} \cdot \frac{1}{z-z_k}.$$

7. Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+2x+2)(x^2+4)}\,dx = -\frac{\pi}{10}$.

8. Given $a > 0$, show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}} e^{-\frac{a}{\sqrt{2}}} \left[\cos\left(\frac{a}{\sqrt{2}}\right) + \sin\left(\frac{a}{\sqrt{2}}\right)\right]$.

9. Show that $\displaystyle\int_{0}^{2\pi} \frac{1}{a+b\sin(\theta)}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$, for $0 < b < a$.

10. Show that $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta)+r^2}\,d\theta = 1$, for $0 \le r < 1$.

    [*Remark*: The integrand $\frac{1-r^2}{1-2r\cos(\theta)+r^2}$ is called the Poisson kernel.]