Math 120A **Homework 1** **Due August 8, 2024**

1. Given a complex number a and a positive real number $\rho > 0$. Show that the equation

$$|z|^2 - 2\,\mathrm{Re}(\bar{a}z) + |a|^2 = \rho^2$$

represents a circle centered at a with radius ρ.

[Hint: The set of points a distance ρ from a satisfy the equation $|z - a| = \rho$.]

2. Given a fixed $a \in \mathbb{C}$. Show that $\dfrac{|z-a|}{|1-\bar{a}z|} = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$

3. Consider the polynomial $p(z) = z^3 + z^2 + z + 1$
   (a) Verify that i is a zero of p.
   (b) Find the other two zeros of p.

4. For which integers n is i an n^{th} root of unity?

5. Let n be an integer with $n \geq 1$.
   (a) Show that $1 + z + z^2 + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}$.
   (b) Show that $1 + \cos(\theta) + \cos(2\theta) + \cdots + \cos(n\theta) = \dfrac{1}{2} + \dfrac{\sin\left[\left(n + \frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{\theta}{2}\right)}$.

6. Show that

$$\frac{(1 + i\tan(\theta))^n}{(1 - i\tan(\theta))^n} = \frac{1 + i\tan(n\theta)}{1 - i\tan(n\theta)}$$

for every integer n.

7. Find the six distinct sixth roots of $z = -64$.

8. Given a complex number $z \in \mathbb{C}$ and the point P on the unit sphere $S \in \mathbb{R}^3$ corresponding to z under stereographic projection. Show that the antipodal point $-P$ corresponds to $-\dfrac{1}{\bar{z}}$ under stereographic projection.

9. Show that a rotation of the unit sphere S by π about the X-axis in $\mathbb{R}^3$ corresponds to the inversion $z \mapsto \dfrac{1}{z}$ of the complex plane $\mathbb{C}$ under stereographic projection.

[Hint: First verify that rotation of the unit sphere S by π about the X-axis in $\mathbb{R}^3$ is represented by $(X, Y, Z) \mapsto (X, -Y, -Z)$.]

10. For each of the following set of points on the unit sphere S in $\mathbb{R}^3$, find an equation that determines the set of corresponding points in the complex plane $\mathbb{C}$ under stereographic projection and describe that set geometrically.
   (a) $\{(X, Y, Z) \in S \mid X^2 + Y^2 = 1\}$.
   (b) $\{(X, Y, Z) \in S \mid X^2 + Z^2 = 1\}$.
   (c) $\{(X, Y, Z) \in S \mid Y^2 + Z^2 = 1\}$.