Math 102 Homework Assignment 3
Due Thursday, February 3, 2022

1. Let $A$ be a $m \times n$ matrix. Show that:
   (a) If $x \in N(A^T A)$, then $Ax \in R(A) \cap N(A^T)$.
   (b) $N(A^T A) = N(A)$.

2. Let $V$ and $W$ be subspaces of $\mathbb{R}^n$ such that $V \subset W$. Show that $W^\perp \subset V^\perp$.

3. Suppose $A$ is a symmetric $n \times n$ matrix. Let $V$ be a subspace of $\mathbb{R}^n$ with the property that $Ax \in V$ for every $x \in V$. Show that $Ay \in V^\perp$ for every $y \in V^\perp$. (Remark: The subspace $V$ is said to be invariant under $A$. This exercise shows that if $V$ is an invariant subspace under $A$, then so is $V^\perp$.)

4. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$.
   (a) Find the orthogonal projection of $b$ onto $R(A)$.
   (b) Describe all least squares solutions to $Ax = b$.

5. Let $A$ be a $n \times n$ matrix. Given that $A I \hat{x} + r = b$, show that $\hat{x}$ is a least squares solution of the system $Ax = b$ and that $r$ is the residual vector.

6. Given a $m \times n$ matrix $A$. Let $\hat{x}$ be a solution to the least squares problem $Ax = b$. Show that a vector $y \in \mathbb{R}^n$ will also be a least squares solution if and only if $y = \hat{x} + z$ for some vector $z \in N(A)$.

7. Let $P_3$ be the inner product space of polynomials of degree less than 3 with inner product $\langle p, q \rangle = \int_{-1}^{1} p(t)q(t) \, dt$. Let $g_1(t) = 1$ and $g_2(t) = t$. Find a basis for $\text{Span}(g_1, g_2)^\perp$, the orthogonal complement of the subspace of $P_3$ spanned by $g_1$ and $g_2$.

8. Let $u$ and $v$ be any two vectors in an inner product space $V$.
   (a) Show that $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$.
   (b) Show that if $\|u - v\|^2 = \|u\|^2 + \|v\|^2$, then $u$ and $v$ are orthogonal.