

Math 142A Homework Assignment 5
Due Tuesday, March 10, 2025

1. Prove that a polynomial $p(x)$ with odd degree has a least one real zero.
2. Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Show that for any $n \in \mathbb{N}$ and points $x_1, x_2, \dots, x_n \in [a, b]$, there is a point $z \in [a, b]$ such that $f(z) = \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$.
3. Suppose that the function $f : [0, 1] \rightarrow \mathbb{R}$ is continuous with $f(0) > 0$, and $f(1) = 0$. Show that there is a number $x_0 \in (0, 1]$ such that $f(x_0) = 0$ and $f(x) > 0$ for every $x \in [0, x_0)$.
4. Let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin\left(\frac{1}{x}\right) & \text{otherwise.} \end{cases}$
Show that f has the intermediate value property on all of $\mathbb{R}$.
5. Suppose that a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ is periodic; that is, there is a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous.
6. Prove that if f is uniformly continuous on a bounded set S , then f is a bounded function on S .
7. Let f be a continuous function on $[0, \infty)$. Prove that if f is uniformly continuous on $[k, \infty)$ for some $k > 0$, then f is uniformly continuous on $[0, \infty)$.
8. Let f be a continuous function on $[a, b]$. Show that the function f^* defined by
$$f^*(x) = \sup \{f(y) \mid a \leq y \leq x\}, \text{ for } x \in [a, b],$$
is an increasing continuous function on $[a, b]$.
9. Suppose the limits $L_1 = \lim_{x \rightarrow a^+} f_1(x)$ and $L_2 = \lim_{x \rightarrow a^+} f_2(x)$ both exist.
 - (a) Show that if $f_1(x) \leq f_2(x)$ for all x in some interval (a, b) , then $L_1 \leq L_2$.
 - (b) Suppose $f_1(x) < f_2(x)$ for all x in some interval (a, b) . Can you conclude that $L_1 < L_2$?
10. Show that if $\lim_{x \rightarrow a^+} f_1(x) = \lim_{x \rightarrow a^+} f_3(x) = L$ and if $f_1(x) \leq f_2(x) \leq f_3(x)$ for all x in some interval (a, b) , then $\lim_{x \rightarrow a^+} f_2(x) = L$. (Note: Be sure to show that $\lim_{x \rightarrow a^+} f_2(x)$ exists, since this is not assumed.)