

Math 142A Homework Assignment 4
Due Tuesday, February 24, 2025

1. Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges.
2. Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information.
3. Let (a_n) be a sequence of nonzero real numbers such that the sequence $\left(\frac{a_{n+1}}{a_n}\right)$ is a constant sequence. Show that $\sum a_n$ is a geometric series.
4. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.
5. (a) Exhibit an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.
(b) Show that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges.
(c) Exhibit an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.
6. Prove that if (a_n) is a decreasing sequence of positive real numbers and if $\sum a_n$ converges, then $\lim_{n \rightarrow \infty} n a_n = 0$.
7. Suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at the point x_0 and that $f(x_0) > 0$. Show that there is an interval $I_n = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ for some $n \in \mathbb{N}$ for which $f(x) > 0$ for every $x \in I_n$.
8. A function $f : D \rightarrow \mathbb{R}$ is said to be *Lipschitz* provided that there is a number $C \geq 0$ with
$$|f(u) - f(v)| \leq C |u - v| \quad \text{for all } u \text{ and } v \text{ in } D.$$
Show that a Lipschitz function is continuous.
9. Suppose the function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ has the property that $\lambda(u+v) = \lambda(u) + \lambda(v)$ for all u, v .
(a) Define the number m by $m := \lambda(1)$. Show that $\lambda(x) = m x$ for all rational numbers x .
(b) Show that if λ is continuous, then $\lambda(x) = m x$ for all $x \in \mathbb{R}$.
10. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.
Show that f is continuous at $x_0 \in \text{dom}(f)$ if and only if for every sequence (x_n) in $\text{dom}(f) \setminus \{x_0\}$ converging to x_0 , we have $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.