

Math 142A Homework Assignment 3
Due 11:00pm Tuesday, February 10, 2025

- (a) Let (s_n) be a monotone sequence. Prove that (s_n) converges if and only if (s_n^2) converges.
(b) Find a non-monotone sequence (t_n) such that (t_n^2) converges but (t_n) does not converge.
- Let (r_n) be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence (r_{n_k}) such that $\lim_{k \rightarrow \infty} r_{n_k} = +\infty$.
- Prove that $\liminf s_n = -\limsup(-s_n)$ for every sequence (s_n) .
[Hint: See Definition 10.6 and Exercise 5.4]
- Let S be a bounded set. Prove that there is an increasing sequence (s_n) in S such that $\lim s_n = \sup(S)$. Explain why if $\sup(S)$ is in S , then it suffices to define $s_n = \sup(S)$ for all n .
- Show that a monotonically increasing sequence is bounded if it has a bounded subsequence.
- Suppose the sequence (s_n) is monotonically increasing and that it has a convergent subsequence. Show that (s_n) converges.
- Let (s_n) and (t_n) be bounded sequences of nonnegative real numbers. Prove that

$$\limsup s_n t_n \leq (\limsup s_n) (\limsup t_n).$$

- Prove the following theorem.

Theorem 1. *Let (s_n) be a bounded sequence. $\liminf s_n = \underline{s}$ if and only if for every $\varepsilon > 0$,*

(i) $s_n > \underline{s} - \varepsilon$ for all but finitely many n , and

(ii) $s_n < \underline{s} + \varepsilon$ for infinitely many n .

- Let (s_n) be a bounded sequence of nonzero real numbers. Prove that

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf |s_n|^{1/n}.$$

- Let (s_n) be a sequence of nonnegative numbers. For each n , define $\sigma_n = \frac{1}{n} (s_1 + s_2 + \cdots + s_n)$. (σ_n) is called the sequence of Cesàro means for (s_n) .
 - Show that $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$.
 - Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.
 - Exhibit an example for which $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.