

Math 142A Homework Assignment 2**Due Tuesday, January 27, 2025**

1. Let (t_n) be a bounded sequence; that is, there exists $M \geq 0$ such that $|t_n| \leq M$ for all n . Let (s_n) be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.
2. Consider three sequences (a_n) , (b_n) , and (s_n) such that $a_n \leq s_n \leq b_n$ for all n , and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.
3. Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove that $\lim s_n = 0$.
4. Suppose that $s_n \neq 0$ for every index n and that the limit $L = \lim \left| \frac{s_{n+1}}{s_n} \right|$ is defined.
 - (a) Show that if $L < 1$, then $\lim s_n = 0$.
 - (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.(See Exercise 9.12 in your text for a hint.)
5. Let $s_1 = 1$, and for $n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$. It turns out that (s_n) converges. Assume this fact and show that $\lim s_n = \frac{1}{2}(1 + \sqrt{5})$.
6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.
 - (a) Show that if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.
 - (b) Does $\lim x_n$ exist? Justify your answer.
 - (c) Explain the apparent contradiction between the result in part (a) and part (b).
7. Show that $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.
8.
 - (a) Verify that $1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$ for $a \neq 1$.
 - (b) Determine $\lim_{n \rightarrow \infty} (1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.
 - (c) What is $\lim_{n \rightarrow \infty} (1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?
9. Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup(S) \notin S$. Show that there is a sequence (s_n) of points in S such that $\lim s_n = \sup(S)$.
10. Let (s_n) be a sequence such that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$.
 - (a) Prove that (s_n) is a Cauchy sequence and, therefore, a convergent sequence.
 - (b) Is (s_n) a Cauchy sequence if we only assume that $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?