

1. Suppose that P and Q are smooth functions on the annulus $\{z \in \mathbb{C} \mid a < |z| < b\}$ that satisfy $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Show directly using Green's theorem that $\oint_{|z|=r} P dx + Q dy$ is independent of the radius r , for $a < r < b$.

2. Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z} dz = 2i \text{Area}(D).$$

3. Suppose $h(z)$ is a continuous function on a curve γ . Show that

$$H(w) = \int_{\gamma} \frac{h(z)}{z - w} dz, \quad w \in \mathbb{C} \setminus \gamma,$$

is analytic on the complement of γ , $\mathbb{C} \setminus \gamma$. Find $H'(w)$.

4. Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_{\gamma} f(z) dz = 0$ for every closed path γ in D .

5. By integrating $e^{-\frac{z^2}{2}}$ around a rectangle $\mathcal{R}_t$ with vertices $-R$, R , $R + it$, $-R + it$, and sending $R \rightarrow \infty$; show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-itx} dx = e^{-\frac{t^2}{2}}, \quad -\infty < t < \infty.$$

Use the known value of the integral for $t = 0$.

6. Suppose $f(z)$ is continuous in the closed disk $\{z \mid |z| \leq R\}$ and analytic on the open disk $\{z \mid |z| < R\}$. Show that $\oint_{|z|=R} f(z) dz = 0$.

[Hint: Approximate $f(z)$ uniformly by $f_r(z) := f(rz)$. Why can't we just use Green's theorem?]

7. Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges.

[Hint: Show that the partial sums of the series satisfy $S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1$.]

8. Show that

(a) $\sum_{k=2}^{\infty} \frac{1}{k \log(k)}$ diverges.

(b) $\sum_{k=2}^{\infty} \frac{1}{k (\log(k))^2}$ converges.

9. Show that $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$.

10. Show that

(a) $\sum_{k=1}^{\infty} \frac{z^k}{k}$ does not converge uniformly for $|z| < 1$.

(b) $\sum_{k=1}^{\infty} \frac{z^k}{k}$ converges uniformly for $|z| \leq \rho$ whenever $0 < \rho < 1$.