

- Let $S \subset \mathbb{C}$ be the set $S = \{z = -it \mid t \geq 0\}$. Define explicitly a continuous branch of $\log(z)$ on $\mathbb{C} \setminus S$; that is, the complex plane slit along the negative imaginary axis. (Hint: Recall that $\text{Log}(z)$, the principal branch of $\log(z)$, is explicitly defined as $\text{Log}(z) = \log|z| + i\theta$, with $-\pi < \theta < \pi$. Be sure to explain why the strict inequalities are required for continuity.)
- Determine the value(s) of $\log[(1+i)^{2i}]$.
- (a) Set $w = \cos(z)$ and $\zeta = e^{iz}$. Show that $\zeta = w + \sqrt{w^2 - 1}$, where $\sqrt{w^2 - 1} = \{\xi \mid \xi^2 = w^2 - 1\}$. How many elements are in $\sqrt{w^2 - 1}$ and how are they related?
(b) Show that

$$\cos^{-1}(w) = -i \log \left[w + \sqrt{w^2 - 1} \right],$$

where both sides of the identity are to be interpreted as subsets of $\mathbb{C}$.

- Let $h : [0, 1] \rightarrow \mathbb{C}$ be a continuous complex-valued function $h(t)$ on the unit interval. Define

$$H(z) := \int_0^1 \frac{h(t)}{t-z} dt, \quad z \in \mathbb{C} \setminus [0, 1]$$

Show that $H(z)$ is analytic on $\mathbb{C} \setminus [0, 1]$ and compute its derivative. (Hint: Differentiate ‘by hand;’ that is, by using the limit definition of the complex derivative.)

- Given an analytic function $f(z) = u(z) + iv(z)$ (with u, v real-valued), we can write z in polar form $z = re^{i\theta}$ and define $u(r, \theta) := u(re^{i\theta})$ and $v(r, \theta) := v(re^{i\theta})$.
(a) Derive the polar form of the Cauchy-Riemann equations for u and v :

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}.$$

(If you are studying a physical science, you should check that this definition yields consistent units across both equations.)

- (b) Check that for any integer m , the functions $u(r, \theta) = r^m \cos(m\theta)$ and $v(r, \theta) = r^m \sin(m\theta)$ satisfy (the polar form of) the Cauchy-Riemann equations.
- Recall that $\cos^{-1}(z) = -i \log \left[z + \sqrt{z^2 - 1} \right]$ (with $\sqrt{z^2 - 1} = \{w \mid w^2 = z^2 - 1\}$). Suppose $g(z)$ is an analytic branch of $\cos^{-1}(z)$, defined on a domain D .
(a) Determine $g'(z)$.
(b) Do different branches of $\cos^{-1}(z)$ have the same derivative?
- Let $f(z)$ be a bounded analytic function, defined on a bounded domain D in the complex plane, and suppose that $f(z)$ is one-to-one. Show that the area of $f(D)$ is given by

$$\text{Area}(f(D)) = \iint_D |f'(z)|^2 dx dy$$

(Hint: Review the change of variables theorem from your vector calculus course.)

- Given a positive number B with $0 < B < \pi$. Find a conformal map of the wedge $\{z \mid -B < \arg(z) < B\}$ onto the right half-plane $\{w \mid \text{Re}(w) > 0\}$.

9. The inversion mapping $f(z) = \frac{1}{z}$ is defined on the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ by defining $f(0) = \infty$, $f(\infty) = 0$, and $f(1) = 1$. (Remember that this does **not** mean that ∞ is a complex number.)

- (a) Show that the image of a straight line under f is a circle or a straight line, depending on whether or not the line passes through the origin.
- (b) Show that the image of a circle under f is a straight line or a circle, depending on whether or not the circle passes through the origin.

Notice that both circles and straight lines in the complex plane $\mathbb{C}$ correspond to circles on the Riemann sphere S .

10. Let $P(x, y)$ and $Q(x, y)$ be continuous complex-valued functions on a curve γ , and define a function $F(w)$ on $\mathbb{C} \setminus \gamma$ by

$$F(w) := \int_{\gamma} \frac{P(x, y)}{z - w} dx + \int_{\gamma} \frac{Q(x, y)}{z - w} dy, \quad \text{where } z = x + iy.$$

- (a) Show that $F(w)$ is analytic for $w \in \mathbb{C} \setminus \gamma$.
- (b) Express $F'(w)$ as a line integral over γ .